PROXIMA project: PROX-Regularity In Mathematical Analysis

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Journées Statistiques Optimisation
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Outline

1. **Proximal-regularity**
   - Few words about PROXIMA project
   - Notion of proximal-regularity
   - Applications in mathematical analysis

2. **Selected challenges**
   - Going beyond prox-regularity
   - From Hilbert setting to Banach spaces
   - Prox-regular programming
   - Smoothness principles

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  - Nonsmooth Analysis
  - Set-Valued Analysis
- V.A. **makes a great use** of:
  - Functional Analysis
  - Measure theory
  - Differential geometry.

Proximal-regularity = Rolling a ball

PROXIMA = Proximal-Regularity In Mathematical Analysis

Let S be a closed subset of a Hilbert space $\mathscr{H}$. One says that S is **r-prox-regular** for a real $r > 0$ if for every $x \in \operatorname{bdry} S$ and every unit (proximal) normal v at x

$$S \cap B(x + rv, r) = \emptyset \quad \big(\text{or equivalently } x \in \operatorname{Proj}_S(x + rv)\big).$$

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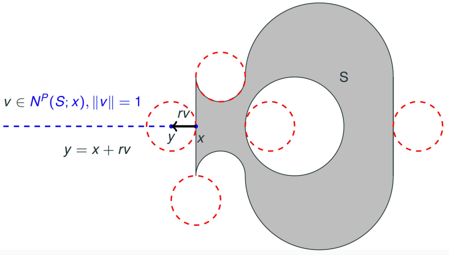

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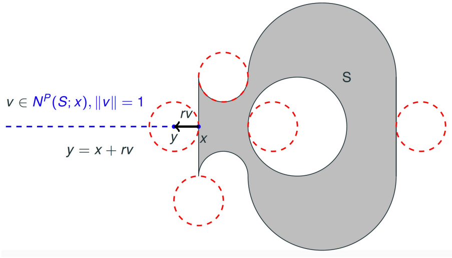
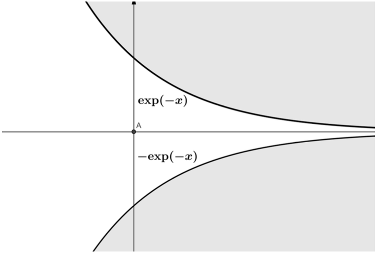


Variants: local prox-regularity, variable radii,...

Variable radii: $\rho(x)$ instead of r, that is $S \cap B\big(x + \rho(x)v, \rho(x)\big) = \emptyset$

Examples and remarks

- Any nonempty closed convex set of $\mathscr{H}$ is $\rho(\cdot)$-prox-regular for any function $\rho : \operatorname{bdry} C \to]0, +\infty[$.
- The complement of the open ball $\mathscr{H} \setminus B(0, r)$ is r-prox-regular.
- The graph of a function $f : \mathscr{H} \to \mathscr{H}$ with a L-Lipschitz Fréchet derivative is L^{-1}-prox-regular.
- $C := \{(x, y) \in \mathbb{R}^2 : y \leq |x|\}$ fails to be $\rho(\cdot)$-prox-regular for any lower semicontinuous function $\rho : \operatorname{bdry} C \to]0, +\infty[$.
- The set $\{(x, y) \in \mathbb{R}^2 : |y| \geq \exp(-x)\}$ is $\rho(\cdot)$-prox-regular (with $\rho(\cdot)$ not bounded from below !).



Let S be a closed subset in a Hilbert space $\mathscr{H}$ and let $r \in]0, +\infty]$.

$$U_r(S) := \{x \in \mathscr{H} : d_S(x) < r\} \text{ and } \frac{1}{r} := 0 \text{ whenever } r = +\infty.$$

Some characterizations of (uniform) prox-regularity

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Theorem

The following assertions are equivalent:

(a) S is r-prox-regular;

(b) For all $x_1, x_2 \in S$, for all $v \in N^P(S;x_1) \cap \mathbb{B}$, $\langle v, x_2 - x_1 \rangle \leq \frac{1}{2r} \|x_2 - x_1\|^2$;

(c) For each $0 < s < r$, the map proj_S is well-defined on $U_s(S)$ and

$$\|\mathrm{proj}_S(u) - \mathrm{proj}_S(v)\| \leq (1 - s/r)^{-1} \|u - v\| \quad \text{for all } u, v \in U_s(S);$$

(d) The function d_S^2 is $C^{1,1}$ on $U_r(S)$;

(e) For all $x_1, x_2 \in S$ and all $t \in [0,1]$ with $tx_1 + (1-t)x_2 \in U_r(S)$,

$$d_S(tx_1 + (1-t)x_2) \leq \frac{1}{2r} \min(t, (1-t)) \|x_1 - x_2\|^2;$$

If in addition S is weakly closed, then one can add:

(f) The mapping $\mathrm{proj}_S(\cdot)$ is continuous on $U_r(S)$.

Comments

- **Prox-regularity has a long story:** G. Durand (1931); N. Aronszajn, K.T. Smith (1956); Yu.G. Reshetnyak (1956); H. Federer (1959); J.-P. Vial (1983); A. Canino (1988); G. Chavent (1991), A. Shapiro (1994); F.H. Clarke, R.L. Stern, P.R. Wolenski (1995); R.A. Poliquin, R. T. Rockafellar, L. Thibault (2000).

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- **Prox-regularity is connected to other classes of sets:**

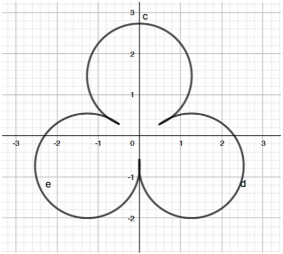
  - **Exterior sphere condition**: for all $x \in \operatorname{bdry} C$, there is $y_x \in \operatorname{bdry} C$ such that $B(y_x, r) \cap C = \emptyset$ and $\|x - y_x\| = r$.
  - **Interior sphere condition**: for all $x \in \operatorname{bdry} C$, there is $y_x \in \operatorname{bdry} C$ such that $B(y_x, r) \subset C$ and $\|x - y_x\| = r$.
  - **Subsmoothness** A set S is subsmooth at $\bar{x} \in S$ provided that

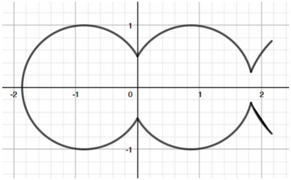
    $$\langle x^\star, x_2 - x_1 \rangle \le \varepsilon \|x_2 - x_1\| \quad \text{for all } x_1, x_2 \in S \cap B(\bar{x}, \delta), x^\star \in N(S; x) \cap \mathbb{B}.$$

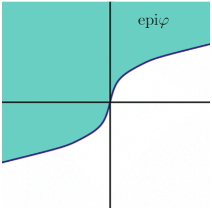
  - **Strong convexity** of radius $R > 0$ = intersection of closed balls with radius R ($\Leftrightarrow$ for all $x, x' \in C$ and all $v \in N(C; x)$ with $\|v\| = 1$,

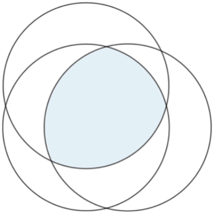
    $$\langle v, x' - x \rangle \le -\frac{1}{2R} \|x' - x\|^2).$$

Examples









Nonconvex functions

Important concepts to go beyond convexity property: *quasiconvexity* (1953), *paraconvexity* (1979), *lower-*C^1 (1981), *lower-*C^2 (1982), *weakly convex functions* (1983), Φ*-convexity* (1983), **primal lower nice/regular** (1991), and **prox-regular functions** *(1996)*, *approximate convex functions* (2000)...

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(Sub)gradient inequality for a (smooth) convex function $f : \mathscr{H} \to \mathbb{R} \cup \{+\infty\}$:

$$f(x') \geq f(x) + \langle \nabla f(x), x' - x \rangle \quad \text{for all } x, x' \in \mathscr{H}$$

- **Prox-regularity** at $\bar{x}$ for $x^\star$ whenever there is $\sigma > 0$ such that

$$f(x') \geq f(x) + \langle x^\star, x' - x \rangle - \sigma \|x' - x\|^2, \tag{1}$$

for all x, x' near $\bar{x}$ with $f(x)$ near $f(\bar{x})$ and $x^\star \in \partial f(x)$.

- **Primal lower-regular/nice** of parameter s

$$f(x') \geq f(x) + \langle x^\star, x' - x \rangle - c(1 + \|x^\star\|)\|x - x'\|^s \tag{2}$$

for appropriate points x, x' and appropriate subgradient $x^\star \in \partial f(x)$.

Some applications of prox-regularity

- Separation properties.
- Differential inclusions/equations (Moreau sweeping process, differential games).
- Various algorithms (projected gradient, alternated projections, averaged projections).
- Metric regularity.
- Determination.
- Extension of Attouch's theorem.
- Selections for multimappings.
- Control (sweeping process, minimum time problem).
- Hamilton-Jacobi.
- Isoperimetric inequalities.

Ball separation property (with a common point)

Theorem (J. Ph. Vial (83), G.E. Ivanov (06))

Let S be an r**-prox-regular** set of the Hilbert space $\mathscr{H}$ with $r > 0$ and C be a non-singleton closed set in $\mathscr{H}$ which is r**-strongly convex** with $C \cap S = \{\bar{x}\}$ and $\bar{x} \in \mathrm{bdry}\, C$.

Then one has the ball separation property for some $v \in \mathscr{H}$ with $\|v\| = 1$

$$B(\bar{x} - rv, r) \cap S = \emptyset \quad \text{and} \quad C \subset B[\bar{x} - rv, r].$$

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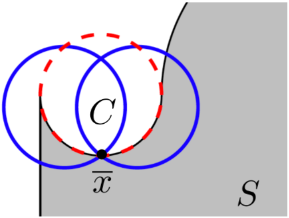

Other separation properties: with $\mathrm{gap}(C,S) > 0$, between a prox-regular set and a point,...

Theorem (Balashov, 22)

Let S be a bounded r-prox-regular set of $\mathbb{R}^N$ and let $f : \mathbb{R}^N \to \mathbb{R}$ be a F-differentiable function. Assume that f (resp. ∇f) is L_0-Lipschitz (resp. L_1-Lipschitz). Let $x_1 \in S$ and $\gamma \in]0, \min(\frac{1}{L_1}, \frac{R}{L_0})[$.

Then, the sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^N$ defined by

$$x_{n+1} := \mathrm{proj}_S\big(x_n - \gamma\nabla f(x_n)\big) \quad \text{for all } n \in \mathbb{N} \tag{3}$$

is well defined.

Optimization under a prox-regular constraint: projected gradient method

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is well defined. Further:

(a) For all $n \in \mathbb{N}$, $f(x_{n+1}) \leq f(x_n) - \frac{1}{2}\big(\frac{1}{\gamma} - L_1\big)\|x_{n+1} - x_n\|$.

(b) One has $\lim_{n\to\infty} d\big(-\nabla f(x_n), N(S;x_n)\big) = 0$.

(c) Every convergent subsequence $(x_{s(n)})_{n\in\mathbb{N}}$ has its limit in the set

$$\Lambda := \{x \in S : -\nabla f(x) \in N(S;x)\}.$$

Differential inclusions

Let $\mathscr{H}$ be a Hilbert space, $C : I = [0,T] \rightrightarrows \mathscr{H}$ be a **nonempty closed convex-valued** multimapping, $u_0 \in C(0)$. In 1971, J.J. Moreau introduced the following **differential inclusion**

$$\begin{cases} -\dot{u}(t) \in N(C(t); u(t)) \quad \lambda\text{-a.e. } t \in I, \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(0) = u_0, \end{cases}$$

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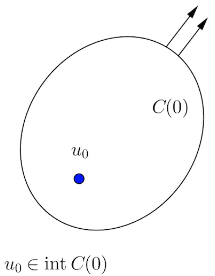

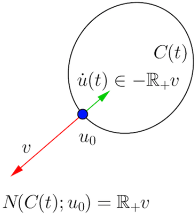

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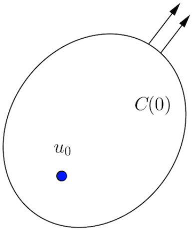
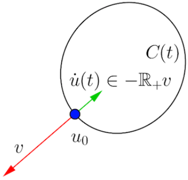


$u_0 \in \operatorname{int} C(0)$



$N(C(t); u_0) = \mathbb{R}_+ v$

Applications: Elastoplasticity, economics, crowd motion, non-regular circuits.

Handling sweeping process: the catching-up algorithm

Step 1: Time discretization $t_i^n := i\frac{T}{2^n}$ and $u_{i+1}^n \in \mathrm{Proj}_{C(t_{i+1}^n)}(u_i^n) \neq \emptyset$.

Step 2: Construction of step mappings

$$I := [0, T] \ni t \mapsto u_n(t) := u_i^n + \frac{t - t_i^n}{t_{i+1}^n - t_i^n}(u_{i+1}^n - u_i^n).$$

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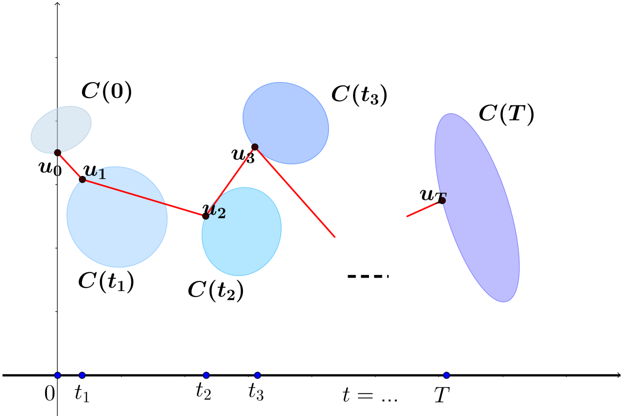

Handling sweeping process: regularization

To solve the differential inclusion

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- **Step 1: Yosida regularization of the normal cone**

$$N(C(t); \cdot) = \partial \psi_{C(t)}(\cdot)$$

is nothing but the **gradient of the Moreau envelope**

$$e_\lambda(\psi_{C(t)})(\cdot) = \frac{1}{2\lambda} d^2_{C(t)}(\cdot).$$

For each $\lambda > 0$, we then consider $u_\lambda(\cdot)$ the unique solution of

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- **Step 2:** Establish $u_\lambda(\cdot) \overset{?}{\to} u(\cdot)$ as $\lambda \downarrow 0$.
- **Step 3:** Show that $u(\cdot)$ is a solution of (SP).

Existence and uniquess of a solution

Theorem (2006)

Assume that there is $r > 0$ such that $C(t) \subset \mathscr{H}$ is r-prox-regular for each t. Assume also that there is an increasing BV function $v : [0,T] \to \mathbb{R}$ such that

$$\mathrm{haus}(C(s), C(t)) \leq v(t) - v(s) \quad \text{for all } s, t.$$

Then, the sweeping process has one and only one solution.

Metric regularity

Prox-regularity is well appropriate to get **metric regularity** property under openness condition.

Theorem (N., Nguyen, Venel (2024))

Let $M : \mathscr{H} \rightrightarrows \mathscr{H}'$ be a multimapping, $\overline{y} \in Y$. Assume that there are two reals $\alpha, \beta > 0$ satisfying $\beta > \frac{1}{2r}(\alpha^2 + \beta^2)$ and such that:

(i) the set $\operatorname{gph} M$ is r-prox-regular;

(ii) $B(\overline{y}, \beta) \subset M(B[\overline{x}, \alpha])$ for all $\overline{x} \in M^{-1}(\overline{y})$.

Then, there exists a real $\gamma \geq 0$ such that for every $\overline{x} \in M^{-1}(\overline{y})$, there exists a real $\delta > 0$ satisfying

$$d\left(x, M^{-1}(\overline{y})\right) \leq \gamma d\left(\overline{y}, M(x)\right) \quad \text{for all } x \in B(\overline{x}, \delta).$$

1. **Proximal-regularity**
   - Few words about PROXIMA project
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2. **Selected challenges**
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   - From Hilbert setting to Banach spaces
   - Prox-regular programming
   - Smoothness principles

Selected challenges

Challenge 1: going beyond prox-regularity

- Replaced usual distance function d_S by some **generalized distance function**

$$\Delta_M(x,y) := d(y, M(x)).$$

Hypomonotonicity property of $(x,y) \mapsto \partial\Delta_M(x,y)$? Nonvacuity property of $\partial\Delta_M(x,y)$? Not so far from prox-regularity of $\operatorname{gph} M$?

- Replaced usual distance function d_S by some **generalized distance function**

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- **Approximate nearest points** ($\alpha = 1,2$)

$$\operatorname{Proj}^{\alpha}_{S,\eta}(x) := \{c \in S : \|x-c\|^{\alpha} \le d^{\alpha}_S(x) + \eta\}. \tag{4}$$

Can we obtained a theory of "approximate prox-regular sets"? At least, investigate some natural properties for approximate nearest points: e.g.,

(a) Lipschitz property: $\operatorname{haus}\left(\operatorname{Proj}^{\alpha}_{S,\eta_1}(x), \operatorname{Proj}^{\alpha}_{S,\eta_2}(x)\right) \le C|\eta_1 - \eta_2|$? This is not so far from transversality property: $d_{S\cap C}(x) \le \alpha\left(d_C(x) + d_S(x)\right)$.

(b) (Co)derivative?

Challenge 2: from Hilbert to Banach setting

Fact. Consider a closed set S in an Hilbert space, $r > 0$, $x \in \operatorname{bdry} S$ and $v \in N(S;x)$ with $\|v\| = 1$. The equivalence

$$S \cap B(x+rv, r) = \emptyset \Leftrightarrow \forall x' \in S, r\|v\|^2 \leq \|x' - (x+rv)\|^2$$

easily leads to the following characterization of r-prox-regularity

$$\langle v, x' - x \rangle \leq \frac{\|v\|}{2r}\|x' - x\|^2 \quad \text{for all } x, x' \in S, v \in N(S;x) \text{ with } \|v\| = 1.$$

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Question. What tool can be used to replace the above square norm development in Banach spaces?

Answer. The duality multimapping:

$$J_p(x) := \left\{ x^\star \in X^\star : \langle x^\star, x \rangle = \|x^\star\|_\star \|x\|, \|x^\star\| = \|x\|^{p-1} \right\} = \partial \frac{1}{p}\|\cdot\|^p(x)$$

Xu-Roach's inequalities

$$\delta(\varepsilon) := \inf\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\}.$$

$$\rho(\tau) := \sup\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = 1 = \|y\|\}$$

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Under some appropriate rotundicity and smoothness properties of the involved norm $\|\cdot\|$, we have the following good behavior (1991, Xu-Roach's inequalities)

$$\langle x^\star - y^\star, x - y\rangle \geq K(\max(\|x\|, \|y\|))^2 \delta\left(\frac{\|x-y\|}{2\max(\|x\|, \|y\|)}\right),$$

where as usual $\delta(\cdot)$ denotes the modulus of uniform convexity of the norm $\|\cdot\|$. In the same line, if the norm $\|\cdot\|$ is uniformly smooth we also have for some $L > 0$

$$\|J_2(x) - J_2(y)\|_\star \leq L(\max(\|x\|, \|y\|))^2 \frac{1}{\|x-y\|}\rho\left(\frac{\|x-y\|}{\max(\|x\|, \|y\|)}\right) \quad \text{for all } x \neq y,$$

Further investigations

- Develop necessary and sufficient conditions for prox-regularity without the help of Xu-Roach's inequalities.
- Prox-regularity with variable radius.
- Preservation of prox-regularity in Banach spaces: provide sufficient condition ensuring the prox-regularity of the constrained set

$$\{x \in \mathscr{H} : g_1(x) \leq 0, \dots, g_m(x) \leq 0, g_{m+1}(x) = 0, \dots, g_{m+n}(x) = 0\}$$

- Going beyond uniform convexity and smoothness of the norm $\|\cdot\|$. Locally uniformly banach spaces ?
- Separation properties, metric regularity,...

Challenge 3: Prox-regular programming

- **"Quasi prox-regularity".** $f : X \to \mathbb{R} \cup \{+\infty\}$ is *quasiconvex* whenever its sublevel sets are convex.

Development of appropriate tools for optimality conditions of quasiconvex functions:

$$N(\operatorname{epi} f; \cdot) \rightsquigarrow N(\{f \le r\}; \cdot)$$

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- **"Maximal hypomonotone operator"**. The normal cone $N(S; \cdot) \cap \mathbb{B}$ to a prox-regular set S enjoys some hypomonotonicity property, namely

$$\langle v_1 - v_2, x_1 - x_2 \rangle \ge -\frac{1}{r} \|x_1 - x_2\|^2 \quad \text{for all } v_i \in N(S; x_i) \cap \mathbb{B}$$

Fact: The normal cone to a nonempty closed convex set is a maximal monotone operator. Can we extend such a result to prox-regular sets?

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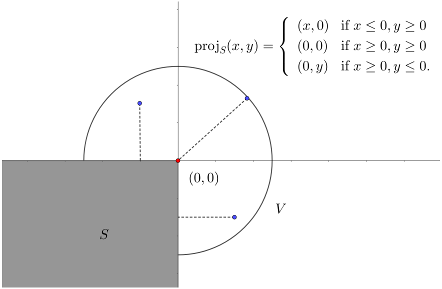
• **"Optimization algorithms"** involving non-monotone operators/prox-regular sets. First work in this direction should be devoted to the classical alternated projection:

$$x_{2n+1} \in \operatorname{Proj}_S(x_{2n}) \quad \text{and} \quad x_{2n+2} \in \operatorname{Proj}'_S(x_{2n+1}).$$

Challenge 4: Differentiability of metric projection

- The convex set $C := \{(x,y) \in \mathbb{R}^2 : x \le 0, y \le 0\}$ **fails** to have its nearest point mapping **differentiable** near 0.



Theorem (Holmes (1973))

Let C be a convex body of a Hilbert space $\mathscr{H}$ whose boundary is a C^{p+1}-submanifold at any of its points. Then, d_C is of class C^{p+1} on $\mathscr{H} \setminus C$ and proj_C is of class C^p on $\mathscr{H} \setminus C$.

Differentiability of metric projection

Theorem (Correa, Salas, Thibault (2018))

Let C be a $\rho(\cdot)$-prox-regular set for some **continuous** function ρ which satisfies some interior tangent cone property at any of its points.

If $\mathrm{bdry}\, C$ is a C^{p+1}-submanifold at any of its points, then d_C (resp. proj_C) is of class C^{p+1} (resp. C^p) on $U_{\rho(\cdot)} \setminus C$.

Corollary: Holmes result for convex bodies. **Converse implication holds** (Salas & Thibault, (2021)): statement (and proof) = technical.

Open question: can we expect something for $\mathrm{dfar}_C(x) := \sup_{c \in C} \|x - c\|$, $\mathrm{far}_C(x)$ where C is a strongly convex set ?

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Thank you for your attention!