Principal Component Analysis for Dependent Functional Data: Incorporating Spatial and Temporal Structures

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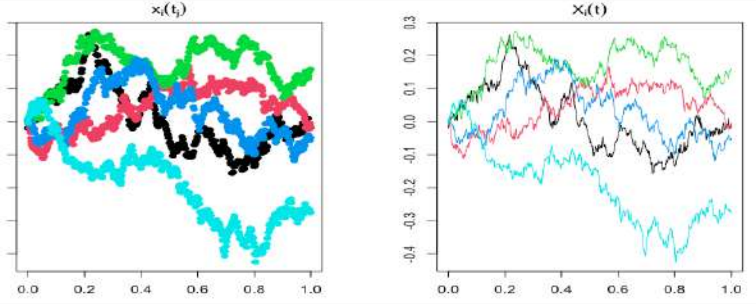
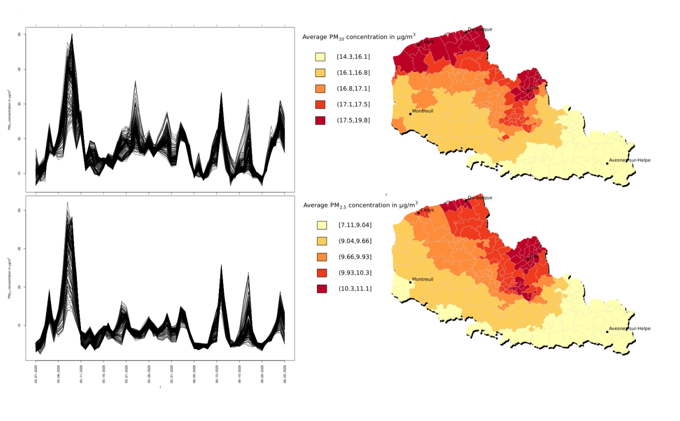
Functional Data

Usual Setting

- X is a random function valued in a space $(\mathcal{F},\ d)$ of eventually infinite dimension.
- $\mathcal{F}$ is typically the space $\mathbb{L}^2$ of square integrable functions defined on some finite interval $\mathcal{D} = [a, b]$.
- n i.i.d. functions $X_1, X_2, \ldots, X_n \sim X$ are observed on $\mathcal{D}$



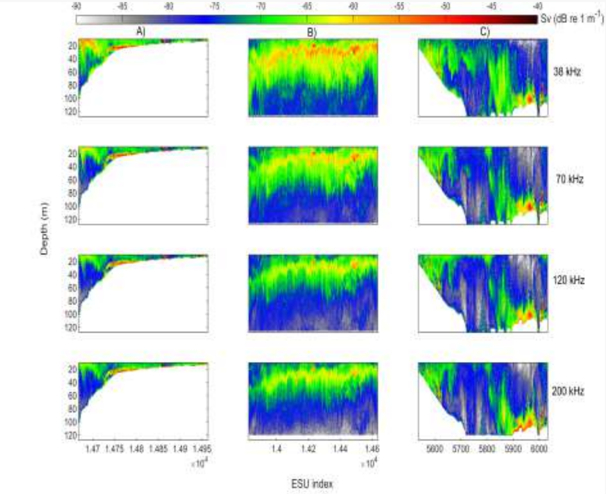
Spatio-temporal pollution data



Source : Frévent, Ahmed, Dabo-Niang, Genin (2023). " Investigating spatial scan statistics for multivariate functional data". JRSS C.

Spatial Acoustic Data (Sv)

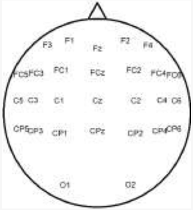
- 2 dimensions : vertically (depths), horizontally (Elementary Sampling Unit ; ESU) in distance (here 0.1 nmi).
- 3 descriptors (depth in meter, thickness in meter, and relative density (mean s_A)) using Matecho.



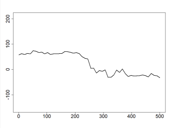
Echogram representing the acoustic intensities

Source : Kande et al. (2024). "Investigating multivariate spatial functional data analysis for acoustic data". Ecological Informatics

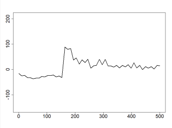
Repeated functional data (Finger Movements)



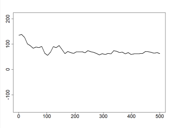
F1 :



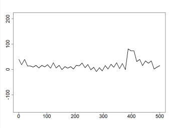
F2 :



F3 :

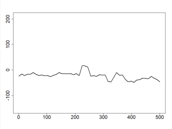


F4 :

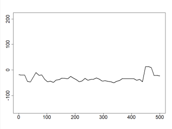


⋮ ⋮

O1 :



O2 :



Source : Moindjie et al. (2025). Fusion regression methods with repeated functional data. CSDA

Why FDA

- Functionality allows broader spectrum of models
  - Estimating model parameters using a single sequence may be limited
  - Time series analysis inherently operates on discrete data, with time stamps assumed to be equally spaced and fixed
- Biological structures are synonymous with Functionality
  - For proteins, the sequence leads to folding (structure), which ultimately determines their function.

Comprehending functions necessitates a grasp of structures.

Structure analysis involves a foundation of mathematical representations followed by the application of probabilistic superstructures.

FDA finds application across all branches of science and engineering.

- **Meteorology/environment** : temperature prediction
- **Computer Vision** : depth sensing, activity recognition, vision-based automation, and the analysis of video data.
- **Computational Biology** : Involves studying complex biomolecular structures and understanding the relationship between organism shapes and functionality.
- **Biometrics and Human Identification** : Includes recognition of human face, body, gait, etc.
- **Wearables, Mobility, Fitness** : Utilized in devices like Fitbit, sleep studies, and motion capture (MoCap) technology.
- **Electricity** : Forecasting electricity consumption.
- Mining, natural sciences, economics, finance, etc

Historical Perspective

An old topic, lots of work already in the past

"FDA" by Jim Ramsay and colleagues in late 1980s

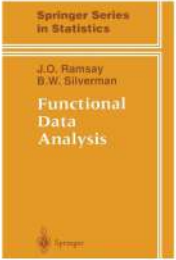
... 1982 1997

WHEN THE DATA ARE FUNCTIONS

J. O. Ramsay

McGill University

Introduction



Celebrating 100 years of the functional linear model

FDA has roots going back to the work of Fisher (1924)

Reproduced from the Philosophical Transactions of the Royal Society, B, 213: 89-142 (1924), with permission of the Royal Society

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III. *The Influence of Rainfall on the Yield of Wheat at Rothamsted.*

By R. A. Fisher, *M.A., Head of the Statistical Department, Rothamsted Experimental Station, Harpenden, Fellow of Gonville and Caius College.*

Communicated by Sir John Russell, *F.R.S.*

(Received September 26,—Read November 8, 1923.)

R.A. Fisher in 1924

Historical Perspective : FDA linear model (Fisher, 1924)

Disregarding, then, both the arithmetical and the statistical difficulties, which a direct attack on the problem would encounter, we may recognise that whereas with q subdivisions of the year, the linear regression equations of the wheat crop upon the rainfall would be of the form

$$w = c + a_1 r_1 + a_2 r_2 + \dots + a_q r_q$$

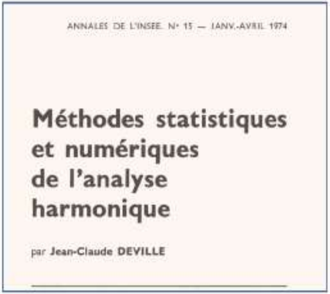
where r_1, r_2, ..., r_q are the quantities of rain in the several intervals of time, and a_1, ... a_q are the regression coefficients, so if infinitely small subdivisions of time were taken, we should replace the linear regression function by a *regression integral* of the form

$$w = c + \int_0^T ar\,dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III)}$$

where $r\,dt$ is the rain falling in the element of time dt ; the integral is taken over the whole period concerned, and a is a *continuous* function of the time t, which it is our object to evaluate from the statistical data.

Thanks to the "Historical FDA elements" by Gilbert Saporta (2024)

Celebrating 50 years of functional PCA



ANNALES DE L'INSEE N° 15 — JANV.-AVRIL 1974

Méthodes statistiques et numériques de l'analyse harmonique

par Jean-Claude DEVILLE

1944-2021

Statistical and numerical methods of harmonic analysis by Deville (1974)

New generation of functional data

Dependency-dimension-structures-nature of sample

- *Shapes, complexe structures, multivariate,...*
- *Non random sample*
- Time/Spatially dependent series :
  everything is related to everything else, but near things are more related than distant things (Tolber, 1970)

New generation of functional data

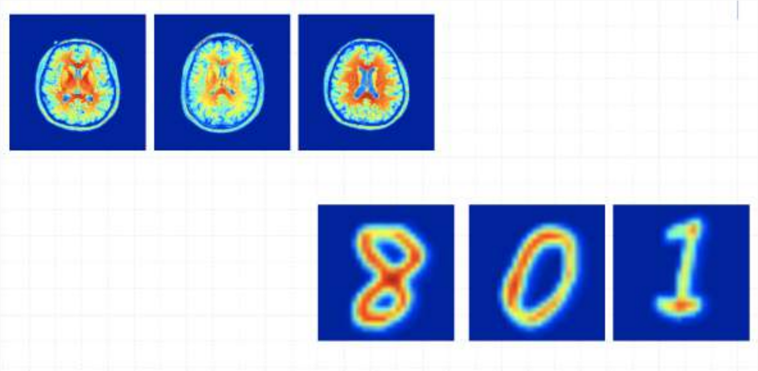
Data as observation of a random variable valued in a (complex) space of functions :

$$\mathbf{X} = \left\{ (X^1(t_1), ..., X^p(t_p))^\top \ : \ t_j \in \mathcal{T}_j, j = 1, ..., p \right\},$$

$$X_{t_j} : \mathcal{P}_j \to \mathcal{S}_j$$

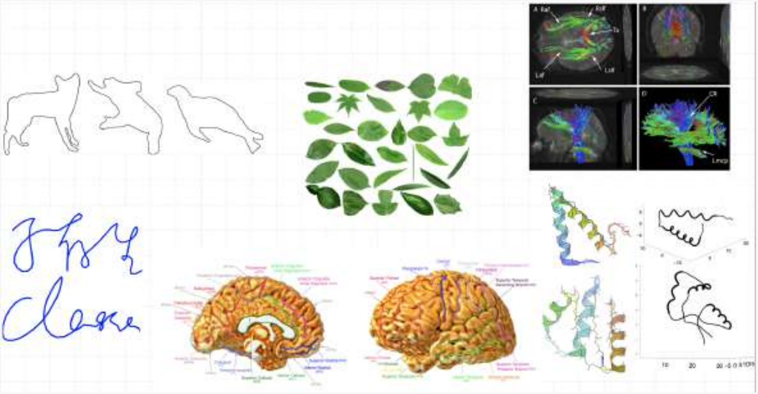
$\mathcal{T}_j \subseteq \mathbb{R}$, $\mathcal{S}_j = \mathbb{R}$ (curve)
$\mathcal{T}_j \subseteq \mathbb{R}$, $\mathcal{S}_j = \{e_1, e_2, \ldots, e_K\}$ (sequence)
$\mathcal{T}_j \subseteq \mathbb{R}^2$, $\mathcal{S}_j = \mathbb{R}$ (image/surface)
...

Multivariate functions, images : $f : [0,1]^2 \to \mathbb{R}^2$



Source : Srivastava

Structures



Source : Srivastava

Gaming, Remote sensing, Mobile depth sensing ...

Source : Srivastava

In this talk

- PCA of time/spatial functional series
- the considered sample is composed of :
  - spatially dependent observations, collected by random sampling process
- specificity of the proposed methods : taking into consideration the sample nature
- applications to regression/classification

FPCA in usual setting and applications to supervised learning

Random Functions Mean and Covariance

X is a random function valued in $\mathbb{L}^2$.

- **Mean function :**

$$\mu(t) = E(X(t))$$

- **Covariance function :**

$$c(t, u) = E\Big((X(t) - \mu(t))(X(u) - \mu(u))\Big)$$

Sample mean, standard deviation and covariance

Pointwise mean :

$$\bar{X}_n(t) = \frac{1}{n} \sum_{i=1}^{n} X_i(t)$$

Pointwise standard deviation :

$$S_n(t) = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} \Big(X_i(t) - \bar{X}_n(t)\Big)^2}$$

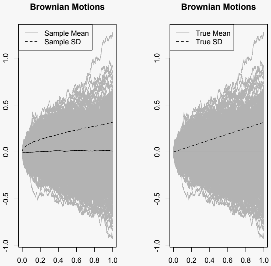
Pointwise covariance function :

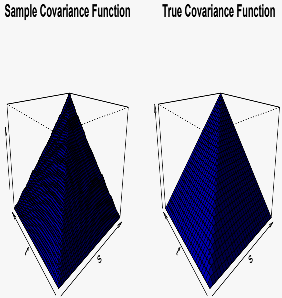
$$\hat{c}_n(t,u) = \frac{1}{n-1} \sum_{i=1}^{n} \Big(X_i(t) - \bar{X}_n(t)\Big)\Big(X_i(u) - \bar{X}_n(u)\Big)$$

- $\bar{X}_n(t)$ and $\hat{c}_n(t,u)$ are **estimators** of the population parameters $\mu(t)$ and $c(t,u)$.
- $\hat{c}_n(t,u)$ is interpreted in a similar way as the usual variance-covariance matrix and is largely used in FDA.

Let $X_1, ..., X_n$ be i.i.d (independent and identically distributed) observations of X

Sample mean, standard deviation and covariance of Brownian Motion





FPCA in usual setting and applications to supervised learning

Modes of variability (PCA)

Covariance function and Principal Component Analysis

Functional Principal Component Analysis (FPCA), allows to represent a square integrable random function X as :

$$X(t) = \mu(t) + \sum_{j=1}^{\infty} \xi_j v_j(t)$$

(**Karhunen-Loéve (KL) expansion**)

- v_j are the **eigenfunctions** and solutions of

$$\int_{\mathcal{D}} c(t,u) v_j(u) du = \lambda_j v_j(t)$$

  .

- $\lambda_1 \geq \lambda_2 \geq \ldots$ are the **eigenvalues**.
- The *random variables* ξ_j are the **scores**

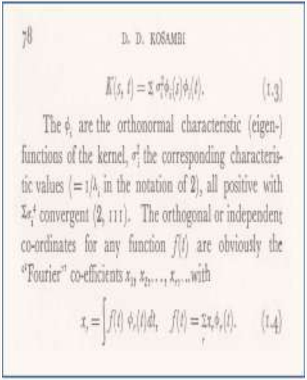
$$\xi_j = \langle X - \mu, v_j \rangle = \int_{\mathcal{D}} (X(t) - \mu(t)) v_j(t) dt$$

- λ_j is the variance of X **in the principal direction** v_j

Karhunen, Loève, Kosambi (KLK) decomposition

The KL deccomposition is commonly attributed to Kari Karhunen (1946) and Michel Loève (1946) but it has been obtained earlier by D.D.Kosambi (1943)

1907-1966



78 D. D. KOSAMBI

$$K(s,t)=\Sigma\sigma_i^2\phi_i(s)\phi_i(t). \quad (1.3)$$

The ϕ_i are the orthonormal characteristic (eigen-) functions of the kernel, σ_i^2 the corresponding characteristic values ($=1/\lambda_i$ in the notation of 2), all positive with $\Sigma\sigma_i^4$ convergent (2, 111). The orthogonal or independent co-ordinates for any function $f(t)$ are obviously the "Fourier" co-efficients $x_1, x_2, \ldots, x_i \ldots$ with

$$x_i=\int f(t)\,\phi_i(t)dt, \quad f(t)=\Sigma_i x_i\phi_i(t). \quad (1.4)$$

FPCA in practice and dimension reduction by FPCA

- Use the Estimated Functional Principal Components (EFPC's) $\hat{v}_j$ as **basis functions** for X_i :

$$X_i(t) \approx \bar{X}_n(t) + \sum_{j=1}^{p_n} \hat{\xi}_{ij} \hat{v}_j(t)$$

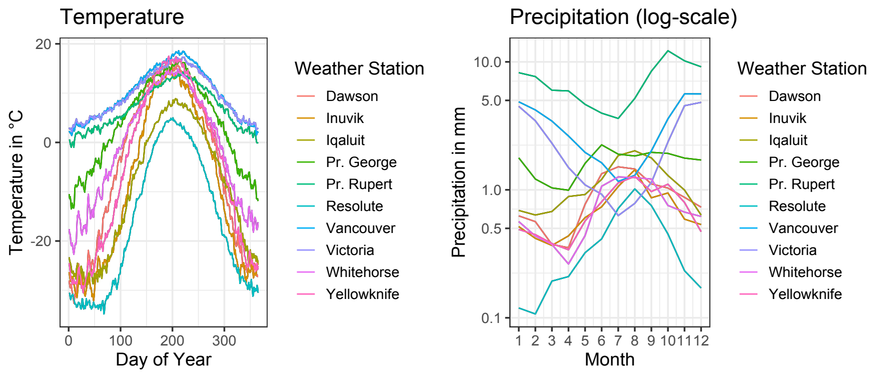
- **Estimated scores** : $\hat{\xi}_{ij} = \int_{\mathcal{D}} (X_i(t) - \bar{X}_n(t)) \hat{v}_j(t) dt$
- EFPC's $\hat{v}_j$ are **orthonormal**, i.e.

$$\int_{\mathcal{D}} \hat{v}_j(t) \hat{v}_k(t) dt = \begin{cases} 1, & j = k \\ 0, & j \neq k. \end{cases}$$

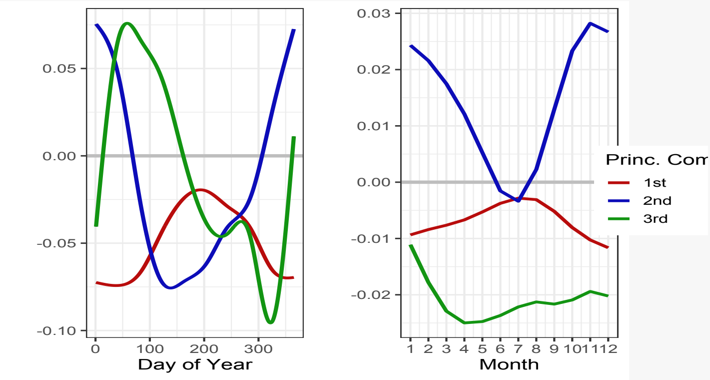
- **Choice of the dimension p_n**

Application to Canadian weather data

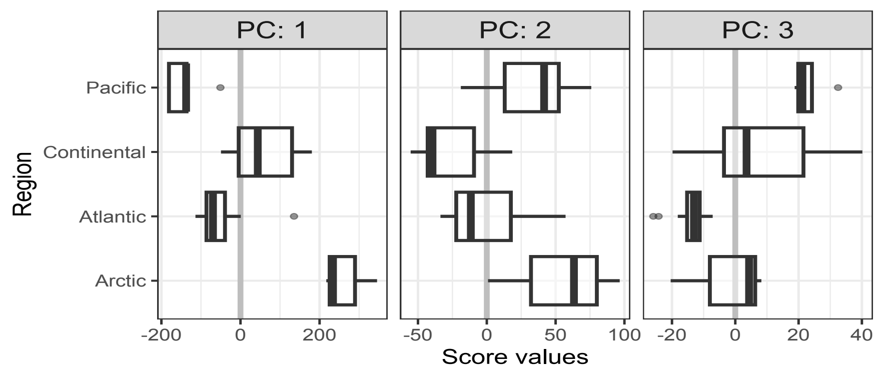
First smooth the data



Principal components (eigen) functions

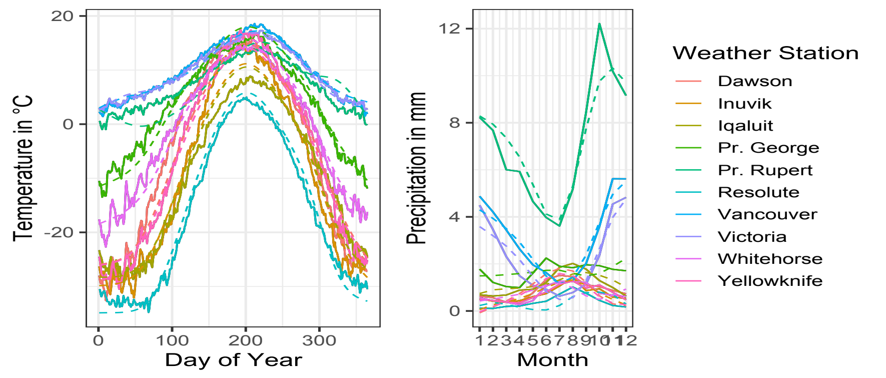


Scores



PC: 1
PC: 2
PC: 3
Region
Pacific
Continental
Atlantic
Arctic
-200
0
200
-50
0
50
100
-20
0
20
40
Score values

Approximation with the first $p = 3$ PCA basis functions

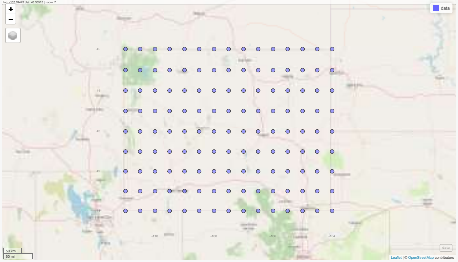


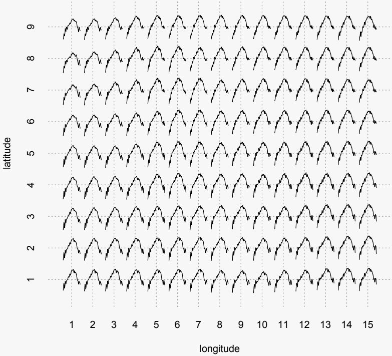
PCA of spatial multivariate functional data

Modeling geospatial functional data

- We consider daily temperature data recorded at n stations from the meteorological monitoring network.
- We have M data at each station corresponding to daily records of maximum temperature obtained from a given period
- Prediction of the whole temperature curve at a given station
- The spatio-temporal dataset could be analyzed by using, space-time geostatistics (space-time kriging, see Cressie and Wikle, 2011).

Geospatial functional data





Modeling spatial functional data

- Modelization of functional data basically focuses on independent data.
- In many applied domains, data are spatially correlated functions :
  economic, environmental, hydrology, ...

  Example : curves of daily concentration of ozone at two near stations
- Some works are developed to deal with spatially correlated functional data
  - Functional geostatistical data :

    PCA and clustering : Kuenzer et al. (2022), Vandewalle et al. (2022), Frevent et. al (2023),...

    PCA and Moran statistics : Assan et al. (2019), Darbi et al (2022) ,...,

    Kriging methods : Monestiez aand Nerini (2008), Giraldo et al. (2010), Bohorquez et al. (2016), ...

    Nonparametric regression : Ternynck (2014), Dabo-Niang et al. (2011, 2018, 2020),...
  - Lattice functional data : less developed

    Ruiz-Medina (2012) : prediction of SAR hilbertian processes

    Pineda-Rios and Giraldo (2016), Zhang et al. (2016) ; Ahmed et al. (2021) ; Huang et al. (2018) : FLMs with SAR disturbance process

Basic notations for functional geo-spatial data

- $X = \left(X_s(.),\ s \in \mathbb{R}^N\right)$, a measurable spatial process $N \geq 1$, defined on some probability space $(\Omega,\ \mathcal{A}, \mathbf{P})$
- X_s is valued in a space $(\mathcal{X},\ d)$ of eventually infinite dimension
- $d(.,.)$ is some measure of proximity, e.g. a metric or a semi-metric
- $\mathcal{X}$ is a space of functions, typically $\mathcal{T} = [0, T]$.
- X is observed at a set of locations $S \subseteq \mathbb{R}^N$ of cardinal n, $S = \{s_1, \ldots, s_n\}$, $s_i \in \mathbb{R}^N$, $i = 1 \ldots n$ and a set of time points $\mathcal{J} = \{t_1, ..., t_M\}$, M
- E the set of the $n \times M$ discrete observations, $E = \{x_{s_i}(t_j), s_i \in S, t_j \in \mathcal{J}\}$.
- Prediction of a whole curve $X_{s_0} = \{X_{s_0}(t),\ t \in \mathcal{T}\}$

Before prediction

- The discrete data $\{x_{s_i}(t), s_i \in S, t \in \mathcal{T}\}$ are converted into curves $\{X_{s_i}(t), s_i \in S, t \in \mathcal{T}\}$ by using smoothing methods (e.g. Splines).
- $\{X_{s_i}(t), s_i \in S, t \in \mathcal{T}\}$ are valued in $\mathcal{X} = L^2[0, T]$
- Expand each $X_{s_i}(.)$ in terms of basis functions (here FPC).
- Take into acount the spatial dimension into the FPCA

Spatial dependence

Weakly stationary functional process

(i) $\mathbb{E}(X_{\mathbf{s}}(t)) = \mathbb{E}(X_{\mathbf{0}}(t)) = \mu(t),\ t \in \mathcal{T}$ does not depend on **s** with **0** the zero vector in $\mathbb{R}^N$

(ii) for all $\mathbf{s}, \mathbf{h} \in S$, and $t, s \in \mathcal{T}$;

$$C_{\mathbf{h}}(t,s) := Cov\Big(X_{\mathbf{h}}(t), X_{\mathbf{0}}(s)\Big) = Cov\Big(X_{\mathbf{s}+\mathbf{h}}(t), X_{\mathbf{s}}(s)\Big)$$

depends only on the spatial lag.

Variogram function

$$2\gamma_{t,t'}(\mathbf{h}) = \mathrm{Var}(X_{\mathbf{s}+\mathbf{h}}(t) - X_{\mathbf{s}}(t'))$$

$$\gamma_t(\mathbf{h}) = \gamma_{t,t}(\mathbf{h})$$

Trace Variogram function

$$\gamma(\mathbf{h}) = \int_{\mathcal{T}} \gamma_t(\mathbf{h})dt$$

$$2\gamma(\mathbf{h}) = E\int_{\mathcal{T}} (X_{\mathbf{s}_i}(t) - X_{\mathbf{s}_j}(t))^2 dt,\ \mathbf{h} = \mathbf{s}_i - \mathbf{s}_j,\ \mathbf{s}_i, \mathbf{s}_j \in S$$

Spectral Spatial FPCA (SFPCA)

Kuenzer et al. (2020), Si-Ahmed et al. (2024).

Let S be a regular grid (rectangular domain) of $\mathbb{Z}^N$, $\mathcal{F}_\theta^X$ be the spectral density operator of $X_{\mathbf{s}}$ with kernel :

$$f_\theta^X(t,s) := \frac{1}{(2\pi)^N} \sum_{\mathbf{h}\in\mathbb{Z}^N} C_{\mathbf{h}}(t,s)\exp(-i\mathbf{h}^\top\theta) \tag{1}$$

$$\mathcal{F}_\theta^X = \sum_{m\geq 1} \lambda_{j,m}(\theta)\varphi_m(\theta)\otimes\varphi_m(\theta) \tag{2}$$

where $\lambda_m(\theta) \geq \lambda_m(\theta) \geq ... \geq 0$

$$\varphi_m(t|\theta) = \sum_{\mathbf{l}\in\mathbb{Z}^N} \phi_{m,\mathbf{l}}(t)\exp(-i\mathbf{l}^\top\theta). \tag{3}$$

The functional principal component score is defined as :

$$\xi_{m,\mathbf{s}} := \sum_{\mathbf{l}\in S} \left\langle X_{\mathbf{s}-\mathbf{l}}, \phi_{m,\mathbf{l}} \right\rangle \tag{4}$$

SFPCA

Karhunen-Loève-Kosambi spatial expansion :

$$X_{\mathbf{s}}(t) = \sum_{m=1}^{\infty} X_{m,\mathbf{s}}(t),\ X_{m,\mathbf{s}}(t) := \sum_{\mathbf{l}\in\mathbb{Z}^N} \xi_{m,\mathbf{s}+\mathbf{l}}\phi_{m,\mathbf{l}}(t), \quad t \in \mathcal{T} \tag{5}$$

The spectral density operator estimate :

$$\widehat{\mathcal{F}}_{\boldsymbol{\theta}}^{X} := \frac{1}{(2\pi)^N} \sum_{\|\mathbf{h}\|\leq \mathbf{q}} w(\mathbf{h}/\mathbf{q})\widehat{C}_{\mathbf{h}} e^{-i\mathbf{h}^\top \boldsymbol{\theta}} \tag{6}$$

$\widehat{C}_{\mathbf{h}}$ the sample autocovariance operators, $w(.)$ a weight function

$$\widehat{C}_{\mathbf{h}} := \frac{1}{n} \sum_{\mathbf{s}\in M_{\mathbf{h},\mathbf{n}}} \left(X_{\mathbf{s}+\mathbf{h}} - \bar{X}\right) \otimes \left(X_{\mathbf{s}} - \bar{X}\right) \tag{7}$$

with $M_{\mathbf{h},\mathbf{n}} = \{\mathbf{s} : 1 \leq \mathbf{s}_i, \mathbf{s}_i + h_i \leq n_i, \forall 1 \leq i \leq N\}$. If the set $M_{\mathbf{h},\mathbf{n}}$ is empty, $\widehat{C}_{\mathbf{h}} = 0$, $n = \prod_{i=1}^{N} n_i$.

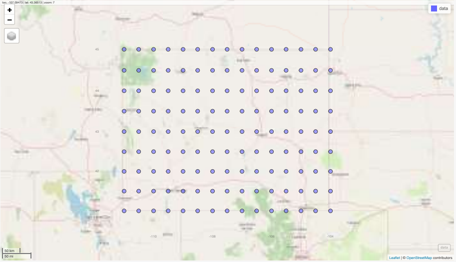
SFPCA

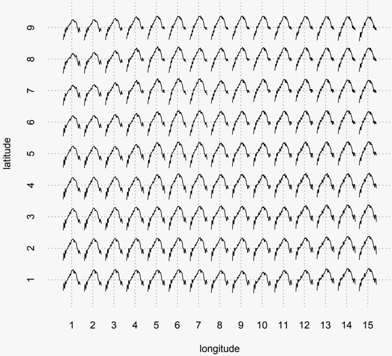
$$X_{\mathbf{s}}(t) \approx \sum_{m=1}^{K} \hat{X}_{m,\mathbf{s}}(t), \quad t \in \mathcal{T},$$

$$\hat{X}_{m,\mathbf{s}}(t) := \sum_{\|\mathbf{l}\|_\infty \leq L} \hat{\xi}_{m,\mathbf{s}+\mathbf{l}} \hat{\phi}_{m,\mathbf{l}},$$

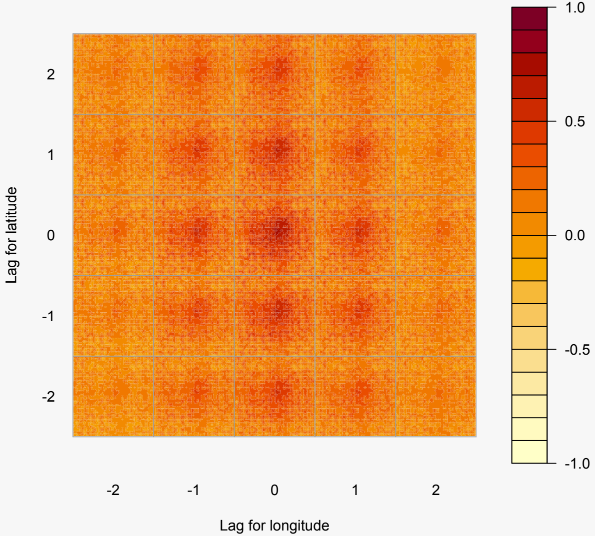
assuming $1 + 2L \leq \mathbf{s}_i \leq n_i - 2L$ for $1 \leq i \leq N$.

Daily temperature data (year 2001)

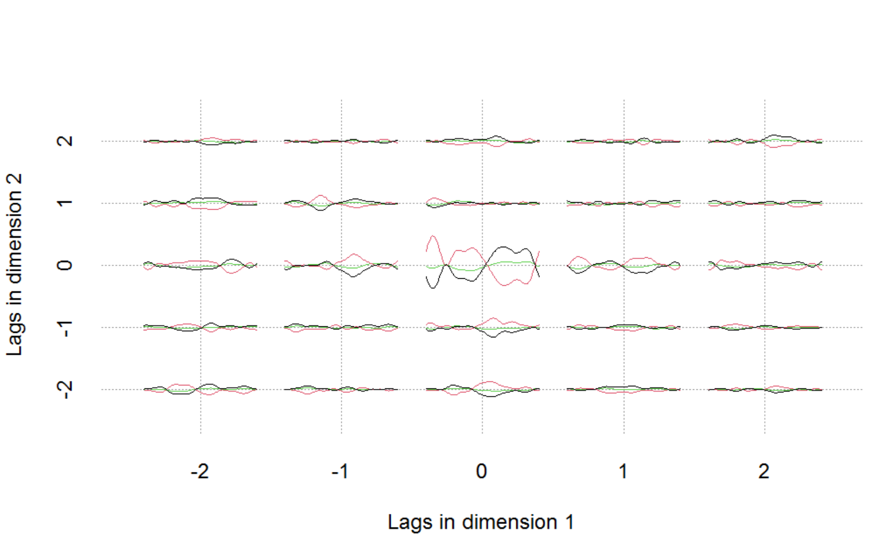




Correlations



Space-time filters



Spectral Principal Component Analysis of Multivariate Spatial Functional Data

Let the covariance operator $C_j := \mathbb{E}[(X^{(j)} - \mu^j) \otimes (X^{(j)} - \mu^j)]$ of X^j

$$(C_j f)(t) = \int_{\mathcal{T}_j} c_j(s,t) f(s) ds, \quad f \in \mathcal{L}^2(\mathcal{T}_j), \; t \in \mathcal{T}_j \tag{8}$$

Weakly stationary functional process

(i) $\mathbb{E}(X_{\mathbf{s}}^{(j)}(t)) = \mathbb{E}(X_{\mathbf{0}}^{(j)}(t)) = \mu^j(t)$, $t \in \mathcal{T}_j$ with $\mathbf{0}$ being the zero vector in $\mathbb{R}^N$

(ii) for all $\mathbf{s}, \mathbf{h} \in \mathbf{D}$, and $t, s \in \mathcal{T}_j$;

$c_{j,\mathbf{h}}(t,s) := Cov\left(X_{\mathbf{h}}^j(t), X_{\mathbf{0}}^j(s)\right) = Cov\left(X_{\mathbf{s+h}}^j(t), X_{\mathbf{s}}^j(s)\right)$

Spectral Principal Component Analysis of Multivariate Spatial Functional Data

Let $\mathcal{F}_{\theta}^{X^{(j)}}$ be the spectral density operator of $X_{\mathbf{s}}^{(j)}$ with the following kernel :

$$f_{\theta}^{X^{(j)}}(t,s) := \frac{1}{(2\pi)^N} \sum_{\mathbf{h}\in\mathbb{Z}^N} c_{j,\mathbf{h}}(t,s) \exp(-i\mathbf{h}^{\top}\theta) \tag{9}$$

$$\mathcal{F}_{\theta}^{X^{(j)}} = \sum_{m\geq 1} \lambda_{j,m}(\theta)\varphi_{j,m}(\theta) \otimes \varphi_{j,m}(\theta) \tag{10}$$

where $\lambda_{j,m}(\theta) \geq \lambda_{j,m}(\theta) \geq ... \geq 0$

$$\varphi_{j,m}(t|\theta) = \sum_{\mathbf{l}\in\mathbb{Z}^N} \phi_{m,\mathbf{l}}^{(j)}(t) \exp(-i\mathbf{l}^{\top}\theta). \tag{11}$$

The functional principal component score is defined as :

$$\xi_{m,\mathbf{s}}^{(j)} := \sum_{\mathbf{l}\in\mathbf{D}} \left\langle X_{\mathbf{s}-\mathbf{l}}^{(j)}, \phi_{m,\mathbf{l}}^{(j)} \right\rangle \tag{12}$$

Spectral Principal Component Analysis of Multivariate Spatial Functional Data (SMFPCA)

The Karhunen-Loève spatial expansion of $X_{\mathbf{s}}^{(j)}$ is given by :

$$X_{\mathbf{s}}^{(j)}(t) = \sum_{m=1}^{\infty} X_{m,\mathbf{s}}^{(j)}(t) \quad t \in \mathcal{T}_j \text{ with} \tag{13}$$

$$X_{m,\mathbf{s}}^{(j)}(t) := \sum_{\mathbf{l} \in \mathbb{Z}^N} \xi_{m,\mathbf{s}+\mathbf{l}}^{(j)} \phi_{m,\mathbf{l}}^{(j)}(t)$$

The spectral density operator is estimated as :

$$\widehat{\mathcal{F}}_{\boldsymbol{\theta}}^{X^{(j)}} := \frac{1}{(2\pi)^N} \sum_{\|\mathbf{h}\| \leq \mathbf{q}} w(\mathbf{h}/\mathbf{q}) \widehat{C}_{j,\mathbf{h}} e^{-i\mathbf{h}^\top \boldsymbol{\theta}} \tag{14}$$

$\widehat{C}_{j,\mathbf{h}}$ the sample autocovariance operators.

SMFPCA Methodology

The multivariate eigenfunctions are :

$$\hat{\psi}^{(j)}_{m,\mathbf{s}}(t_j) \approx \sum_{l=1}^{M_j} [\hat{c}_m]^{(j)}_l \, \hat{\phi}^{(j)}_{l,\mathbf{s}}(t_j) \tag{15}$$

$$t_j \in \mathcal{T}_j, \mathbf{s} \in \mathbf{D}, \;\; m = 1, ..., M_+$$

Multivariate PCA scores

$$\hat{\rho}_{m,\mathbf{s}} = \sum_{j=1}^{p} \sum_{l=1}^{M_j} [\hat{c}_m]^{(j)}_l \, \hat{\xi}^{(j)}_{l,\mathbf{s}} \tag{16}$$

$$X^{(j)}_{\mathbf{s}}(t_j) \approx \sum_{m=1}^{M_j} \hat{X}^{(j)}_{m,\mathbf{s}}(t_j), \quad t_j \in \mathcal{T}_j, \text{ with } \quad \hat{X}^{(j)}_{m,\mathbf{s}}(t_j) := \sum_{\|\mathbf{l}\|_\infty \leq L} \hat{\xi}^{(j)}_{m,\mathbf{s}+\mathbf{l}} \hat{\phi}^{(j)}_{m,\mathbf{l}} \tag{17}$$

Spatial MFPCA compare to MFPCA (Happ and Greven (2018))

$$\text{NMSE}(M_j) = \frac{\sum_{\mathbf{s}\in\mathbf{D_n}} \left\| X_{\mathbf{s}}^{(j)} - \sum_{m=1}^{M_j} \widehat{X}_{m,\mathbf{s}}^{(j)} \right\|^2}{\sum_{\mathbf{s}\in\mathbf{D_n}} \left\| X_{\mathbf{s}}^{(j)} \right\|^2} \tag{18}$$

$\mathbf{D_n} = \{\mathbf{s} \in \mathbb{Z}^N : 1 \leq \mathbf{s}_i \leq n_i \text{ for all } 1 \leq i \leq n\})$ represents a region where the mean is calculated

$$\text{NMSE}^*_{\text{spat}}(M_j) = 1 - \frac{\sum_{m\leq M_j} \int_{[-\pi,\pi]^N} \hat{\lambda}_{j,m}(\boldsymbol{\theta})d\boldsymbol{\theta}}{\sum_{m\geq 1} \int_{[-\pi,\pi]^N} \hat{\lambda}_{j,m}(\boldsymbol{\theta})d\boldsymbol{\theta}} \tag{19}$$

Spatial MFPCA compare to MFPCA (Happ and Greven (2018))

1. NMSE and NMSE^* results obtained by SMFPCA and MFPCA with 2 functional time series (2000, 2001).

Cumulative PCA	PC1		PC2		PC3	
Spatial consideration	Spatial	Ordinary	Spatial	Ordinary	Spatial	Ordinary
NMSE 2000	0.4796	0.5416	0.3396	0.5147	0.2103	0.3749
NMSE^* 2000	0.4356	0.5156	0.2596	0.3342	0.1664	0.2695
NMSE 2001	0.5178	0.6016	0.3665	0.4121	0.3578	0.3627
NMSE^* 2001	0.5061	0.6021	0.2709	0.3788	0.1678	0.2686

2. NMSE and NMSE^* results obtained by SMFPCA and MFPCA considering 3 series $(1996, 1998, 1999)$

Cumulative PCA	PC1		PC2		PC3	
Spatial consideration	Spatial	Ordinary	Spatial	Ordinary	Spatial	Ordinary
NMSE 1996	0.5090	0.6364	0.5069	0.5215	0.3223	0.5029
NMSE^* 1996	0.4523	0.5358	0.2786	0.3772	0.1794	0.2851
NMSE 1998	0.6980	0.7111	0.3418	0.5812	0.3026	0.5069
NMSE^* 1998	0.4476	0.5791	0.2624	0.3855	0.1640	0.2837
NMSE 1999	0.4377	0.4762	0.3053	0.3941	0.2758	0.3237
NMSE^* 1999	0.4254	0.4744	0.2739	0.3520	0.1889	0.2778

Functional Kriging : Prediction of the whole temperature curve at a given station

Variogram Estimation

Let us suppose an isotropic variogram : $\gamma(\mathbf{h}) = \gamma(\|\mathbf{h}\|)$

The trace-variogram estimate is

$$\hat{\gamma}_n(\mathbf{h}) = \frac{1}{2\#N(\mathbf{h})} \sum_{\mathbf{s}_i, \mathbf{s}_j \in N(\mathbf{h})} \int_{\mathcal{T}} (X_{\mathbf{s}_i}(t) - X_{\mathbf{s}_j}(t))^2 dt,$$

where $N(\mathbf{h}) = \{(\mathbf{s}_i, \mathbf{s}_j) \; : \; h - \Delta \leq \|\mathbf{s}_i - \mathbf{s}_j\| \leq \mathbf{h} + \Delta \, ; \quad i, j = 1, \ldots, n\}$.

Ordinary functional Kriging

$$\hat{X}_{\mathbf{s}_0} = \sum_{i=1}^{n} \lambda_i X_{\mathbf{s}_i}$$

$$E(\hat{X}_{\mathbf{s}_0}) = E(X_{\mathbf{s}_0}) \text{ and } E \int_{\mathcal{T}} (\hat{X}_{\mathbf{s}_0}(t) - X_{\mathbf{s}_0}(t))^2 dt \text{ minimum}$$

The $(\lambda_i)_{i=1,n}$ are solutions of the system (m is a Lagrange multiplier)

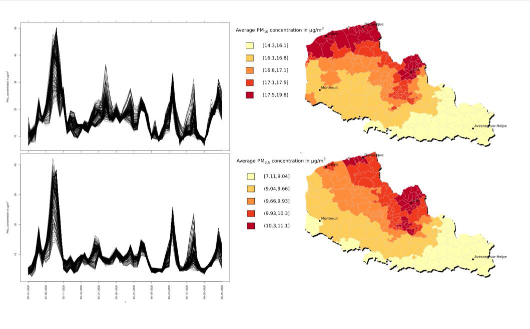
$$\begin{pmatrix} 0 & \gamma(\|\mathbf{s}_1 - \mathbf{s}_2\|) & \dots & \gamma(\|\mathbf{s}_1 - \mathbf{s}_n\|) & 1 \\ \gamma(\|\mathbf{s}_1 - \mathbf{s}_2\|) & 0 & \dots & \gamma(\|\mathbf{s}_2 - \mathbf{s}_n\|) & 1 \\ \dots & \dots & \dots & \dots & \\ \gamma(\|\mathbf{s}_1 - \mathbf{s}_n\|) & \gamma(\|\mathbf{s}_2 - \mathbf{s}_n\|) & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \dots \\ \lambda_n \\ m \end{pmatrix} = \begin{pmatrix} \gamma(\|\mathbf{s}_0 - \mathbf{s}_1\|) \\ \gamma(\|\mathbf{s}_0 - \mathbf{s}_2\|) \\ \dots \\ \gamma(\|\mathbf{s}_0 - \mathbf{s}_n\|) \\ 1 \end{pmatrix}$$

Kriging Variance

$$\sigma^2_{OK}(\mathbf{s}_0) = E((\hat{X}_{\mathbf{s}_0} - X_{\mathbf{s}_0})^2) = m + \sum_{i=1}^{n} \lambda_i \gamma(\|\mathbf{s}_i - \mathbf{s}_0\|)$$

Other dependencies

Areal data Spatial Functional PCA



Source : (Pathmanathan et al. 2024). "Spatial Principal Component Analysis and Moran Statistics for Multivariate Functional Areal Data". Under review.

- Generalized functional dynamic PCA : (Khoo et al. 2024).

Thank you for listening