On computing upper bounds for nonlinear min problems involving disjunctive constraints: applications

Sonia Cafieri

ENAC - École Nationale de l'Aviation Civile

Université de Toulouse, France

RÉPUBLIQUE FRANÇAISE
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Journées de statistique et optimisation en Occitanie

UP VD Université Perpignan Via Domitia

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Focus on **nonlinear constrained problems** having a **strucure**

often arising in applications

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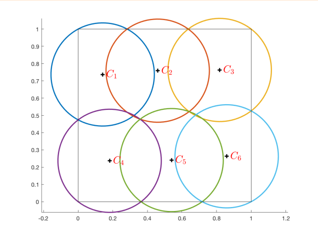
What do the problems below have in common?



covering a rectangle by circles whose radius is to minimize

keeping a safety separation between pairs of aircraft

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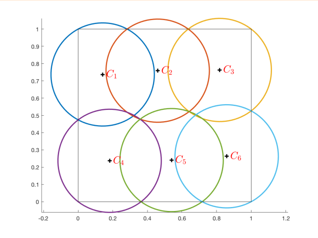

covering a rectangle by circles whose radius is to minimize

keeping a safety separation between pairs of aircraft

mathematical optimization formulation involving disjunctive constraints

→ the *only* combinatorial aspect

Focus on **nonlinear problems** whose *only* combinatorial aspect comes from **disjunctive constraints**

Constraints of the form: $t(x) > 0 \Rightarrow f(x) \geq 0$ logically equivalent to

$$t(x) \leq 0 \text{ or } f(x) \geq 0$$

- common in mathematical optimization models
- typically modelled by introducing auxiliary binary variables

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In this talk: Approach relying on the

continuous quadrant penalty formulation of disjunctive constraints

as a continuous-optimization alternative to the mixed-integer formulations

continuous nonconvex formulation

yields an efficient computation of upper bounds to be used in B&B-based approaches

Outline

1. Continuous penalty-based formulation of logical constraints
2. An application from discrete geometry
3. An application from Air Traffic Management
4. Conclusions and perspectives

Outline

1 Continuous penalty-based formulation of logical constraints

2 An application from discrete geometry

3 An application from Air Traffic Management

4 Conclusions and perspectives

Logical constraints: Typical formulations

$$\boxed{t(x) \le 0 \quad \text{or} \quad f(x) \ge 0}$$

introducing a binary variable z:

Big-M formulation

$$\begin{aligned} t(x) &\le M_1 z \\ -f(x) &\le M_2(1-z) \\ z &\in \{0,1\} \end{aligned}$$

where M_1 and M_2 **large enough** so that:

$t(x) \le M_1$ and $-f(x) \le M_2$

for all desirable solutions x

⇒ potential numerical instability/inefficiency

Complementary formulation

$$\begin{aligned} t(x)(1-z) &\le 0, \\ f(x)\,z &\ge 0, \\ z &\in \{0,1\} \end{aligned}$$

⇒ nonlinear constraints

A continuous-optimization alternative

Introduced in:

S.C., A.R. Conn, M. Mongeau,
The continuous quadrant penalty formulation of logical constraints.
Open Journal on Mathematical Optimization, 2023

Using penalty functions to model logical constraints

Intuition: guide the search of a continuous-optimization method towards the parts of the domain where the logical constraint is satisfied

Reformulating a logical constraint

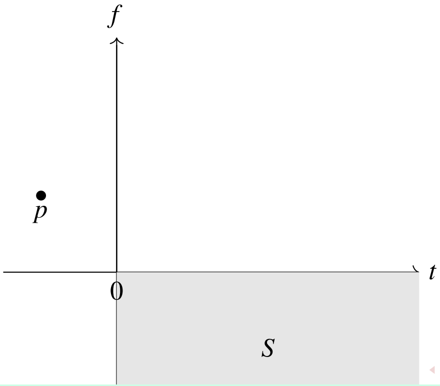
Let $x \in \mathbb{R}^n$, and consider $(t(x), f(x)) \in \mathbb{R}^2$ (in the sequel, we drop the dependency upon x)

Requiring $\boxed{t \leq 0 \quad \text{or} \quad f \geq 0}$

is equivalent to

requiring $\boxed{p := (t,f) \in \mathbb{R}^2 \setminus S}$

where $S := \{(t,f) : t > 0 \text{ and } f < 0\}$ (the open **fourth quadrant is forbidden**)



What type of function?

To guide the search so as $p := (t, f)$ is driven outwards from S (= the 4th quadrant), we would like a function $g : p \in \mathbb{R}^2 \to \mathbb{R}$ satisfying:

a) $g(p) = 0$, if $p \in \mathbb{R}^2 \setminus S$ and $g(p) > 0$, if $p \in S$

b) g **leans outwards** S
i.e., if for any given point $\bar{p} \in S$, and for any descent direction $\bar{d}$ for g at $\bar{p}$, there exists a threshold step size $\bar{\gamma} > 0$ such that $\bar{p} + \gamma\bar{d} \notin S$, for all $\gamma \geq \bar{\gamma}$
[a descent method minimizing g converges towards a point in $\mathbb{R}^2 \setminus S$]

c) g is continuous

d) g is smooth

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[a descent method minimizing g converges towards a point in $\mathbb{R}^2 \setminus S$*]*

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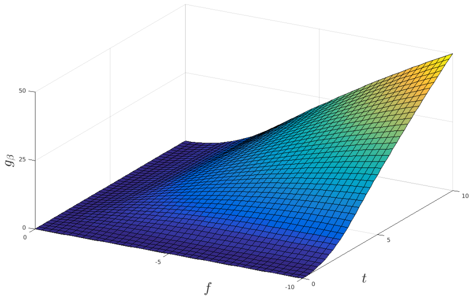


Why this function? Search for a penalty function g

Consider some forbidden set: $\emptyset \neq S \subsetneq \mathbb{R}^n$.

A linear function?

Proposition (S.C., A.R.Conn, M.Mongeau, OJMO 2023)

Let $S \subset \mathbb{R}^n$ be such that (the desirable set) $\mathbb{R}^n \setminus S$ is not convex.
Then, no function $g : \mathbb{R}^n \to \mathbb{R}$ s.t. $g(p) = 0$, if $p \in \mathbb{R}^n \setminus S$, and $g(p) > 0$, if $p \in S$, can be convex

$\Longrightarrow$ g cannot be linear (was rather obvious)

A piecewise-linear function?

Proposition (S.C., A.R.Conn, M.Mongeau, OJMO 2023)

Unless S is a half-space,
*$\nexists$ a continuous **two-piece** piecewise-linear function $g : \mathbb{R}^n \to \mathbb{R}$ s.t.*
$g(p) = 0$, if $p \in \mathbb{R}^n \setminus S$, and $g(p) > 0$, if $p \in S$.

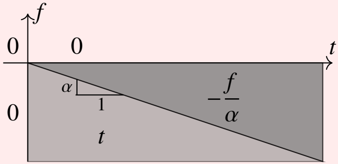
$\Longrightarrow$ at least 3 pieces needed

Back to our context: $n = 2$ and S = the open 4th quadrant.

Proposition (S.C., A.R.Conn, M.Mongeau, OJMO 2023)

*There exists a continuous piecewise-linear function $g : \mathbb{R}^2 \to \mathbb{R}$ s.t. $g(p) = 0$, if $p \in \mathbb{R}^2 \setminus S$, and $g(p) > 0$, if $p \in S$, with **three pieces**:*

$$g_\alpha(t,f) = \begin{cases} 0, & \text{if} \quad t \le 0 \ \text{or} \ f \ge 0 \\ -\dfrac{f}{\alpha}, & \text{if} \quad -\alpha\, t \le f \le 0 \\ t, & \text{if} \quad 0 \le t \le -\dfrac{f}{\alpha}, \end{cases}$$



where $\alpha > 0$ is any given positive (slope) parameter.
*Moreover, up to a multiplicative constant, and up to the arbitrary value of α, this function is **unique**.*

changing the units of t and f (scaling) $\leftrightarrow$ changing the slope α $\implies$ set $\alpha = 1$

Proposition (S.C., A.R.Conn, M.Mongeau, OJMO 2023)

g_α satisfies:
a) $g_\alpha(p) = 0$, if $p \in \mathbb{R}^n \setminus S$ and $g_\alpha(p) > 0$, if $p \in S$ b) g_α leans outwards S c) g_α is continuous

... but g_α is **not smooth**

A smooth piecewise-quadratic penalty function

Let $S \subseteq \mathbb{R}^2$ be the open fourth quadrant.

Proposition (S.C., A.R.Conn, M.Mongeau, OJMO 2023)

*There exists a penalty function, $g : \mathbb{R}^2 \to \mathbb{R}$, that is **piecewise quadratic** with exactly **4 pieces**, satisfying:*

a) $g(p) = 0$, *if* $p \in \mathbb{R}^2 \setminus S$ *and* $g(p) > 0$, *if* $p \in S$

b) g *leans outwards* S

c) g *is continuous*

d) g *is smooth*

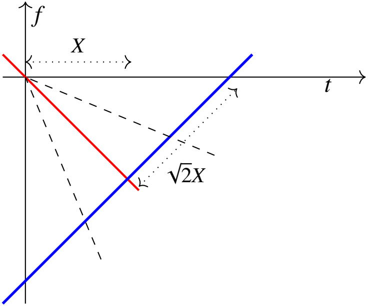
A family of such functions is:

$$g_\beta(t, f) = \begin{cases} 0 & \text{if} \quad t \leq 0 \;\; \text{or} \;\; f \geq 0 \\ t^2 & \text{if} \quad 0 < t \leq -\dfrac{f}{\beta} \\ \dfrac{1}{1-\beta^2}(t^2 + 2\beta\, t\, f + f^2) & \text{if} \quad -\dfrac{f}{\beta} < t < -\beta f \\ f^2 & \text{if} \quad -\dfrac{t}{\beta} \leq f < 0 \end{cases}$$

for $\beta \in \mathbb{R}$, $\beta > 1$.

Restrictions of g_β when $\beta = 3$

For instance choosing $\beta = 3$
(equally-spreaded breakpoints for the bell-shaped curve h_X)



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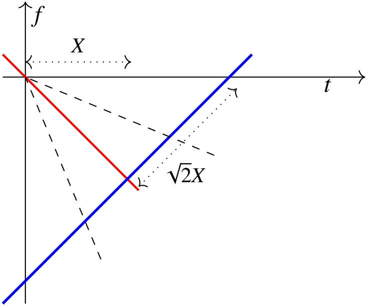

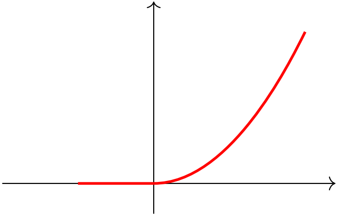

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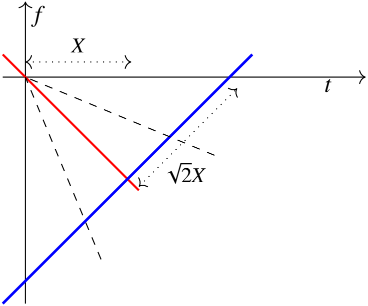
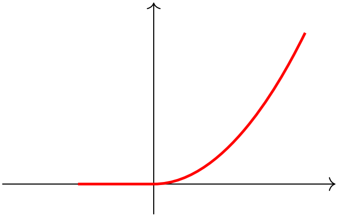
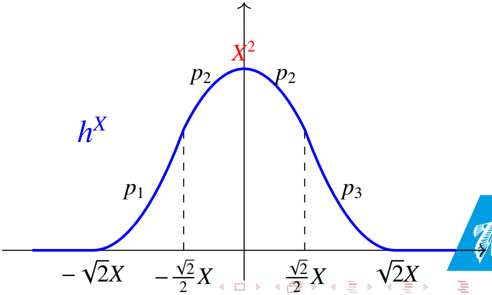
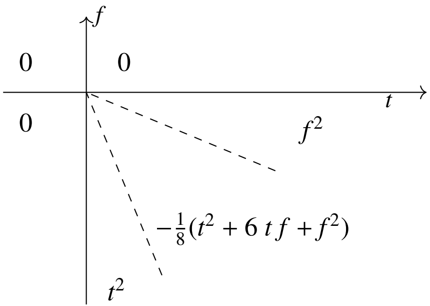






g_β when $\beta = 3$

$$g_\beta(t,f) = \begin{cases} 0 & \text{if} \quad t \le 0 \ \text{ or } \ f \ge 0 \\ t^2 & \text{if} \quad 0 < t \le -\frac{f}{3} \\ -\frac{1}{8}(t^2 + 6\,t\,f + f^2) & \text{if} \quad -\frac{f}{3} < t < -3f \\ f^2 & \text{if} \quad -\frac{t}{3} \le f < 0. \end{cases}$$



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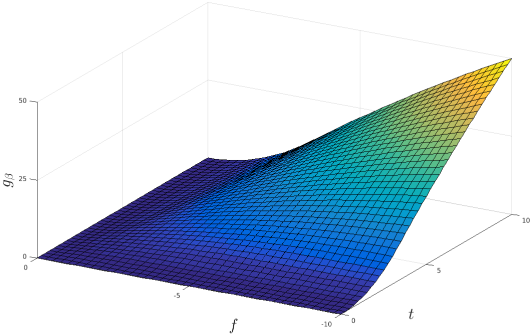

Using g_β

g_β allows using state-of-the-art solvers for nonlinear continuous optimization even in presence of disjunctions

- g_β non convex function $\Rightarrow$ **local** optima
- possible convergence to a local min violating the (penalized) logical constraints (*local infeasibility*)

But good properties $\rightarrow$ yields good-quality **upper bounds**

(to be used in Branch-and-Bound)

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Covering problem

How can a **rectangle** be covered by exactly $n = \mathbf{6}$ **identical circles**, minimizing the radius of the circles?

Melissen and Schuur's conjecture (2000): **circle configurations**

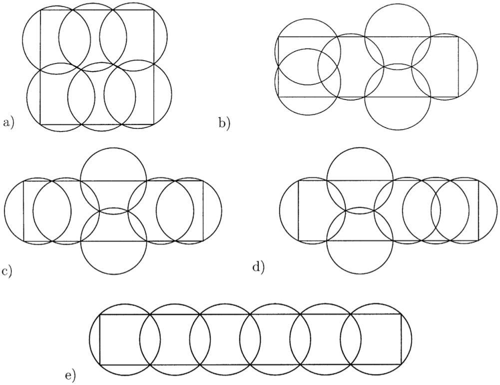


Fig. 3. Thin(nest) coverings of a rectangle with six circles for $1 \leqslant a \leqslant 2.923\ldots$ (a), $2.923\ldots \leqslant a \leqslant 3.118\ldots$ (b), $3.118\ldots \leqslant a \leqslant 3.464\ldots$ (c and d), and $a \geqslant 3.464\ldots$ (e).

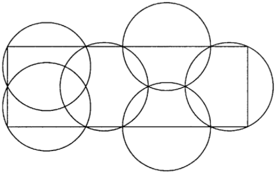
Literature: $n = 6$ circles

$$a = \text{side length of the rectangle (the other side length is 1)}$$
$$r(a) = \text{minimum radius of 6 identical circles covering the rectangle}$$

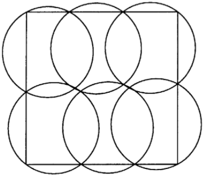
Analytical expressions of $r(a)$ known for configurations c), d), e).

Recently closed cases in:

S. Cafieri, P. Hansen, F. Messine,
Global exact optimization for covering a rectangle with 6 circles.
Journal of Global Optimization, 83, 2022.



- configuration b): $a \in [2.923, 3.118]$
  expression of $r(a)$

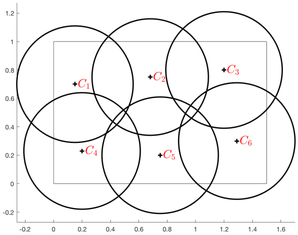


- configuration a): $a \in [1, 2.923]$
  MINLP formulation
  $\rightarrow$ numerical (globally) optimal solutions

Mathematical optimization formulation

Decision variables

- r, radius of the circles
- (x_i, y_i), $\forall i = 1, \ldots, 6$,
  coordinates of the centers of circles C_i
  in an Euclidean space



Objective function

r, to be minimized

Constraints

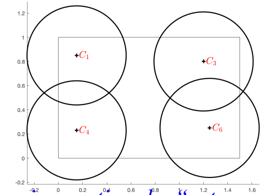
ensuring that the circles are placed in such a way that the rectangle is entirely covered

1. covering the rectangle vertices
2. covering the rectangle sides
3. covering the rectangle interior

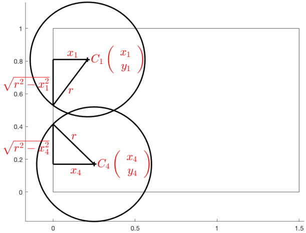
Mathematical optimization formulation

Configuration a): $a \in [1,\ 2.923]$

$$
\left\{
\begin{array}{lcl}
\displaystyle\min_{x_i, y_i, r} & & r \\
s.t. & & \\
(x_1 - 0)^2 + (y_1 - 1)^2 & \leq & r^2 \\
(x_4 - 0)^2 + (y_4 - 0)^2 & \leq & r^2 \\
(x_6 - a)^2 + (y_6 - 0)^2 & \leq & r^2 \\
(x_3 - a)^2 + (y_3 - 1)^2 & \leq & r^2 \\
 & & \\
y_1 - \sqrt{r^2 - x_1^2} & \leq & y_4 + \sqrt{r^2 - x_4^2} \\
y_3 - \sqrt{r^2 - (1 - x_3)^2} & \leq & y_6 + \sqrt{r^2 - (1 - x_6)^2} \\
x_2 - \sqrt{r^2 - (1 - y_2)^2} & \leq & x_1 + \sqrt{r^2 - (1 - y_1)^2} \\
x_3 - \sqrt{r^2 - (1 - y_3)^2} & \leq & x_2 + \sqrt{r^2 - (1 - y_2)^2} \\
x_5 - \sqrt{r^2 - y_5^2} & \leq & x_4 + \sqrt{r^2 - y_4^2} \\
x_6 - \sqrt{r^2 - y_6^2} & \leq & x_5 + \sqrt{r^2 - y_5^2}
\end{array}
\right.
$$



covering vertices by "external" circles



covering the rectangle sides

Mathematical optimization formulation

Configuration a): $a \in [1,\ 2.923]$

$$
\left\{
\begin{array}{lcl}
\min\limits_{x_i, y_i, r} & & r \\
s.t. & & \\
(x_1 - 0)^2 + (y_1 - 1)^2 & \leq & r^2 \\
(x_4 - 0)^2 + (y_4 - 0)^2 & \leq & r^2 \\
(x_6 - a)^2 + (y_6 - 0)^2 & \leq & r^2 \\
(x_3 - a)^2 + (y_3 - 1)^2 & \leq & r^2 \\
 & & \\
y_1 - \sqrt{r^2 - x_1^2} & \leq & y_4 + \sqrt{r^2 - x_4^2} \\
y_3 - \sqrt{r^2 - (1 - x_3)^2} & \leq & y_6 + \sqrt{r^2 - (1 - x_6)^2} \\
x_2 - \sqrt{r^2 - (1 - y_2)^2} & \leq & x_1 + \sqrt{r^2 - (1 - y_1)^2} \\
x_3 - \sqrt{r^2 - (1 - y_3)^2} & \leq & x_2 + \sqrt{r^2 - (1 - y_2)^2} \\
x_5 - \sqrt{r^2 - y_5^2} & \leq & x_4 + \sqrt{r^2 - y_4^2} \\
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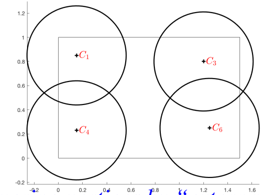
$$



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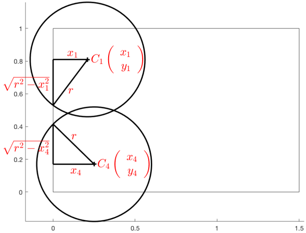

covering the rectangle sides

Mathematical optimization formulation

$$
\begin{cases}
(x_{14}-x_1)^2+(y_{14}-y_1)^2 = r^2 \\
(x_{14}-x_4)^2+(y_{14}-y_4)^2 = r^2 \\
(x_{36}-x_3)^2+(y_{36}-y_3)^2 = r^2 \\
(x_{36}-x_6)^2+(y_{36}-y_6)^2 = r^2 \\
(x_{25}^{[1]}-x_2)^2+(y_{25}^{[1]}-y_2)^2 = r^2 \\
(x_{25}^{[1]}-x_5)^2+(y_{25}^{[1]}-y_5)^2 = r^2 \\
(x_{25}^{[2]}-x_2)^2+(y_{25}^{[1]}-y_2)^2 = r^2 \\
(x_{25}^{[2]}-x_5)^2+(y_{25}^{[2]}-y_5)^2 = r^2 \\
z_{14}\left((x_{14}-x_2)^2+(y_{14}-y_2)^2-r^2\right) \le 0 \\
(1-z_{14})\left((x_{14}-x_5)^2+(y_{14}-y_5)^2-r^2\right) \le 0 \\
z_{25}^{[1]}\left((x_{25}^{[1]}-x_1)^2+(y_{25}^{[1]}-y_1)^2-r^2\right) \le 0 \\
(1-z_{25}^{[1]})\left((x_{25}^{[1]}-x_4)^2+(y_{25}^{[1]}-y_4)^2-r^2\right) \le 0 \\
z_{25}^{[2]}\left((x_{25}^{[2]}-x_3)^2+(y_{25}^{[2]}-y_3)^2-r^2\right) \le 0 \\
(1-z_{25}^{[2]})\left((x_{25}^{[2]}-x_6)^2+(y_{25}^{[2]}-y_6)^2-r^2\right) \le 0 \\
z_{36}\left((x_{36}-x_2)^2+(y_{36}-y_2)^2-r^2\right) \le 0 \\
(1-z_{36})\left((x_{36}-x_5)^2+(y_{36}-y_5)^2-r^2\right) \le 0 \\
0 \le r \le 1,\quad 0 \le x_i \le a,\quad 0 \le y_i \le 1, \quad z_{jk} \in \{0,1\}
\end{cases}
$$

covering the interior of the rectangle:

- *intersection points* (x_{jk}, y_{jk}) *of two circles* C_j, C_k
- *disjunctions:* (x_{jk}, y_{jk}) *have to belong to one of the neighbor circles*

Mathematical optimization formulation

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z_{14}\left((x_{14}-x_2)^2+(y_{14}-y_2)^2-r^2\right) \le 0\\
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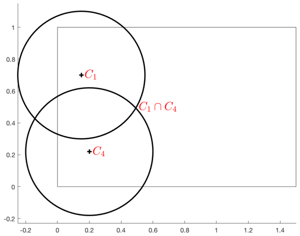
z_{36}\left((x_{36}-x_2)^2+(y_{36}-y_2)^2-r^2\right) \le 0\\
(1-z_{36})\left((x_{36}-x_5)^2+(y_{36}-y_5)^2-r^2\right) \le 0\\
0 \le r \le 1,\ \ 0 \le x_i \le a,\ \ 0 \le y_i \le 1, \qquad z_{jk} \in \{0,1\}
\end{cases}
$$

covering the interior of the rectangle:

- *intersection points* (x_{jk}, y_{jk}) *of two circles* C_j, C_k
- *disjunctions:* (x_{jk}, y_{jk}) *have to belong to one of the neighbor circles*

Example:
(x_{14}, y_{14}) *must belong to* C_2 *or* C_5



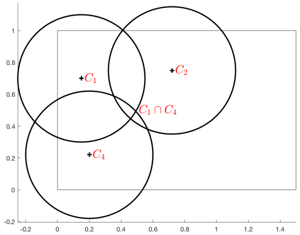
Mathematical optimization formulation

$$
\begin{cases}
(x_{14}-x_1)^2+(y_{14}-y_1)^2 = r^2 \\
(x_{14}-x_4)^2+(y_{14}-y_4)^2 = r^2 \\
(x_{36}-x_3)^2+(y_{36}-y_3)^2 = r^2 \\
(x_{36}-x_6)^2+(y_{36}-y_6)^2 = r^2 \\
(x_{25}^{[1]}-x_2)^2+(y_{25}^{[1]}-y_2)^2 = r^2 \\
(x_{25}^{[1]}-x_5)^2+(y_{25}^{[1]}-y_5)^2 = r^2 \\
(x_{25}^{[2]}-x_2)^2+(y_{25}^{[1]}-y_2)^2 = r^2 \\
(x_{25}^{[2]}-x_5)^2+(y_{25}^{[2]}-y_5)^2 = r^2 \\
z_{14}\left((x_{14}-x_2)^2+(y_{14}-y_2)^2-r^2\right) \le 0 \\
(1-z_{14})\left((x_{14}-x_5)^2+(y_{14}-y_5)^2-r^2\right) \le 0 \\
z_{25}^{[1]}\left((x_{25}^{[1]}-x_1)^2+(y_{25}^{[1]}-y_1)^2-r^2\right) \le 0 \\
(1-z_{25}^{[1]})\left((x_{25}^{[1]}-x_4)^2+(y_{25}^{[1]}-y_4)^2-r^2\right) \le 0 \\
z_{25}^{[2]}\left((x_{25}^{[2]}-x_3)^2+(y_{25}^{[2]}-y_3)^2-r^2\right) \le 0 \\
(1-z_{25}^{[2]})\left((x_{25}^{[2]}-x_6)^2+(y_{25}^{[2]}-y_6)^2-r^2\right) \le 0 \\
z_{36}\left((x_{36}-x_2)^2+(y_{36}-y_2)^2-r^2\right) \le 0 \\
(1-z_{36})\left((x_{36}-x_5)^2+(y_{36}-y_5)^2-r^2\right) \le 0 \\
0 \le r \le 1, \quad 0 \le x_i \le a, \quad 0 \le y_i \le 1, \qquad z_{jk} \in \{0,1\}
\end{cases}
$$

covering the interior of the rectangle:

- *intersection points* (x_{jk}, y_{jk}) *of two circles* C_j, C_k
- *disjunctions:* (x_{jk}, y_{jk}) *have to belong to one of the neighbor circles*

Example:
(x_{14}, y_{14}) *must belong to* C_2 *or* C_5



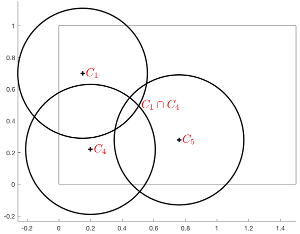
Mathematical optimization formulation

$$
\left\{
\begin{array}{lcl}
(x_{14}-x_1)^2+(y_{14}-y_1)^2 &=& r^2\\
(x_{14}-x_4)^2+(y_{14}-y_4)^2 &=& r^2\\
(x_{36}-x_3)^2+(y_{36}-y_3)^2 &=& r^2\\
(x_{36}-x_6)^2+(y_{36}-y_6)^2 &=& r^2\\
(x_{25}^{[1]}-x_2)^2+(y_{25}^{[1]}-y_2)^2 &=& r^2\\
(x_{25}^{[1]}-x_5)^2+(y_{25}^{[1]}-y_5)^2 &=& r^2\\
(x_{25}^{[2]}-x_2)^2+(y_{25}^{[1]}-y_2)^2 &=& r^2\\
(x_{25}^{[2]}-x_5)^2+(y_{25}^{[2]}-y_5)^2 &=& r^2\\
z_{14}\left((x_{14}-x_2)^2+(y_{14}-y_2)^2-r^2\right) & & \leq 0\\
(1-z_{14})\left((x_{14}-x_5)^2+(y_{14}-y_5)^2-r^2\right) & & \leq 0\\
z_{25}^{[1]}\left((x_{25}^{[1]}-x_1)^2+(y_{25}^{[1]}-y_1)^2-r^2\right) & & \leq 0\\
(1-z_{25}^{[1]})\left((x_{25}^{[1]}-x_4)^2+(y_{25}^{[1]}-y_4)^2-r^2\right) & & \leq 0\\
z_{25}^{[2]}\left((x_{25}^{[2]}-x_3)^2+(y_{25}^{[2]}-y_3)^2-r^2\right) & & \leq 0\\
(1-z_{25}^{[2]})\left((x_{25}^{[2]}-x_6)^2+(y_{25}^{[2]}-y_6)^2-r^2\right) & & \leq 0\\
z_{36}\left((x_{36}-x_2)^2+(y_{36}-y_2)^2-r^2\right) & & \leq 0\\
(1-z_{36})\left((x_{36}-x_5)^2+(y_{36}-y_5)^2-r^2\right) & & \leq 0\\
0\leq r\leq 1,\ \ 0\leq x_i\leq a,\ \ 0\leq y_i\leq 1, & & z_{jk}\in\{0,1\}
\end{array}
\right.
$$

covering the interior of the rectangle:

- *intersection points (x_{jk}, y_{jk}) of two circles C_j, C_k*
- *disjunctions:* *(x_{jk}, y_{jk}) have to belong to one of the neighbor circles*

Example:
(x_{14}, y_{14}) *must belong to* C_2 *or* C_5



Formulation based on the penalty function g_β

disjunctions

$$\left.\begin{array}{ll} z_{14}\left((x_{14}-x_2)^2+(y_{14}-y_2)^2-r^2\right) & \le 0 \\ (1-z_{14})\left((x_{14}-x_5)^2+(y_{14}-y_5)^2-r^2\right) & \le 0 \end{array}\right\}$$

$$\left.\begin{array}{ll} z_{25}^{[1]}\left((x_{25}^{[1]}-x_1)^2+(y_{25}^{[1]}-y_1)^2-r^2\right) & \le 0 \\ (1-z_{25}^{[1]})\left((x_{25}^{[1]}-x_4)^2+(y_{25}^{[1]}-y_4)^2-r^2\right) & \le 0 \end{array}\right\}$$

$$\left.\begin{array}{ll} z_{25}^{[2]}\left((x_{25}^{[2]}-x_3)^2+(y_{25}^{[2]}-y_3)^2-r^2\right) & \le 0 \\ (1-z_{25}^{[2]})\left((x_{25}^{[2]}-x_6)^2+(y_{25}^{[2]}-y_6)^2-r^2\right) & \le 0 \end{array}\right\}$$

$$\left.\begin{array}{ll} z_{36}\left((x_{36}-x_2)^2+(y_{36}-y_2)^2-r^2\right) & \le 0 \\ (1-z_{36})\left((x_{36}-x_5)^2+(y_{36}-y_5)^2-r^2\right) & \le 0 \end{array}\right\}$$

reformulated introducing:

$$g_\beta^1(t^1,f^1) \quad \begin{array}{l} t^1 = \left((x_{14}-x_2)^2+(y_{14}-y_2)^2-r^2\right) \\ f^1 = \left((x_{14}-x_5)^2+(y_{14}-y_5)^2-r^2\right) \end{array}$$

$$g_\beta^2(t^2,f^2) \quad \begin{array}{l} t^2 = \left((x_{25}^{[1]}-x_1)^2+(y_{25}^{[1]}-y_1)^2-r^2\right) \\ f^2 = \left((x_{25}^{[1]}-x_4)^2+(y_{25}^{[1]}-y_4)^2-r^2\right) \end{array}$$

$$g_\beta^3(t^3,f^3) \quad \begin{array}{l} t^3 = \left((x_{25}^{[2]}-x_3)^2+(y_{25}^{[2]}-y_3)^2-r^2\right) \\ f^3 = \left((x_{25}^{[2]}-x_6)^2+(y_{25}^{[2]}-y_6)^2-r^2\right) \end{array}$$

$$g_\beta^4(t^4,f^4) \quad \begin{array}{l} t^4 = \left((x_{36}-x_2)^2+(y_{36}-y_2)^2-r^2\right) \\ f^4 = \left((x_{36}-x_5)^2+(y_{36}-y_5)^2-r^2\right) \end{array}$$

Changing the objective to:

$$\min_{x_i,y_i,r} \; r + \lambda^1 g_\beta^1(t^1,f^1) + \lambda^2 g_\beta^2(t^2,f^2) + \lambda^3 g_\beta^3(t^3,f^3) + \lambda^4 g_\beta^4(t^4,f^4)$$

with $\lambda^1,\lambda^2,\lambda^3,\lambda^4 \ge 0$ penalty parameters

and keeping all the other constraints

$\Longrightarrow$ (nonconvex) NLP reformulation

Numerical results

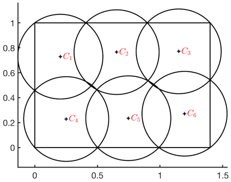
AMPL model implementation, MINLP solver: `COUENNE 0.5`, NLP solver: `IPOPT 3.12`
2.66 GHz, 32 GB RAM

data a	r^*
1.0	0.29873
1.1	0.30808
1.2	0.31803
1.3	0.32853
1.4	0.33954
1.5	0.35099
1.6	0.36287
1.7	0.37512
1.8	0.38771
1.9	0.40060
2.0	0.41377
2.1	0.42720
2.2	0.44085
2.3	0.45471
2.4	0.46876
2.5	0.48298
2.6	0.49736
2.7	0.51189
2.8	0.52654
2.9	0.54132

NLP reformulation with g^1_β, g^2_β, g^3_β, g^4_β solved by `IPOPT`

$\Longrightarrow$ locally optimal solutions
Time (s) < 0.03 for all instances

Example of solution: $a = 1.4$



Numerical results

AMPL model implementation, MINLP solver: `COUENNE 0.5`, NLP solver: `IPOPT 3.12`
2.66 GHz, 32 GB RAM

data a	r^*
1.0	0.29873
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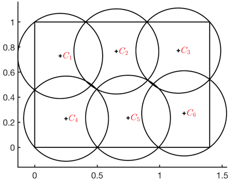
2.4	0.46876
2.5	0.48298
2.6	0.49736
2.7	0.51189
2.8	0.52654
2.9	0.54132

NLP reformulation with g_β^1, g_β^2, g_β^3, g_β^4 solved by `IPOPT`

$\Longrightarrow$ locally optimal solutions
Time (s) < 0.03 for all instances

Example of solution: $a = 1.4$



Numerical results

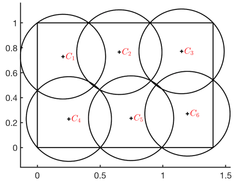
AMPL model implementation, MINLP solver: `COUENNE 0.5`, NLP solver: `IPOPT 3.12`
2.66 GHz, 32 GB RAM

data a	r^*	MINLP Time(s)	MINLP nodes
1.0	0.29873	1.01	12
1.1	0.30808	1.05	12
1.2	0.31803	1.06	14
1.3	0.32853	0.79	8
1.4	0.33954	0.76	6
1.5	0.35099	0.77	8
1.6	0.36287	0.73	6
1.7	0.37512	0.92	30
1.8	0.38771	0.85	26
1.9	0.40060	0.84	24
2.0	0.41377	**2.11**	664
2.1	0.42720	0.88	86
2.2	0.44085	**2.65**	908
2.3	0.45471	1.62	482
2.4	0.46876	0.64	6
2.5	0.48298	**6.63**	3170
2.6	0.49736	**9.65**	4490
2.7	0.51189	0.79	8
2.8	0.52654	1.00	30
2.9	0.54132	**40.9**	19970

NLP reformulation with g_β^1, g_β^2, g_β^3, g_β^4 solved by `IPOPT`

$\Longrightarrow$ locally optimal solutions
Time (s) < 0.03 for all instances

Example of solution: $a = 1.4$



Numerical results

AMPL model implementation, MINLP solver: `COUENNE 0.5`, NLP solver: `IPOPT 3.12`
2.66 GHz, 32 GB RAM

data a	r^*	MINLP Time(s)	MINLP nodes	MINLP + UB-NLP Time(s)	MINLP + UB-NLP nodes
1.0	0.29873	1.01	12	0.89	10
1.1	0.30808	1.05	12	1.41	18
1.2	0.31803	1.06	14	0.87	6
1.3	0.32853	0.79	8	0.86	6
1.4	0.33954	0.76	6	0.77	6
1.5	0.35099	0.77	8	0.74	6
1.6	0.36287	0.73	6	0.83	6
1.7	0.37512	0.92	30	0.80	28
1.8	0.38771	0.85	26	0.71	6
1.9	0.40060	0.84	24	1.29	21
2.0	0.41377	**2.11**	664	**0.62**	6
2.1	0.42720	0.88	86	0.93	58
2.2	0.44085	**2.65**	908	**0.75**	8
2.3	0.45471	1.62	482	1.04	90
2.4	0.46876	0.64	6	0.92	8
2.5	0.48298	**6.63**	3170	**0.77**	6
2.6	0.49736	**9.65**	4490	**0.95**	46
2.7	0.51189	0.79	8	0.79	6
2.8	0.52654	1.00	30	1.02	92
2.9	0.54132	**40.9**	19970	**0.78**	10

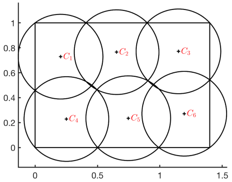
NLP reformulation with g_β^1, g_β^2, g_β^3, g_β^4 solved by `IPOPT`

$\Longrightarrow$ locally optimal solutions
Time (s) < 0.03 for all instances

$\Longrightarrow$ upper bound as an artificial cutoff to `COUENNE` $\longrightarrow$ `UB-NLP`

Note: needs setting penalty parameters

Example of solution: $a = 1.4$



Outline

1. Continuous penalty-based formulation of logical constraints
2. An application from discrete geometry
3. An application from Air Traffic Management
4. Conclusions and perspectives

Aircraft conflict avoidance

- n aircraft, $A := \{1, 2, \ldots, n\}$
- straight-line segment trajectories
- Given, $\forall i \in A$:
  - initial position, $(x_i^0, y_i^0) \in \mathbb{R}^2$
  - initial velocity
    (heading angle, ϕ_i, and speed, v_i)

Aim: decide changes of **heading angles** and **speeds** at $t = 0$
to ensure pairwise aircraft separation:

$$\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \geq d \qquad \textit{for all } t \geq 0$$

s.t. **bound** constraints (**feasibility** problem)

Inter-distance safety separation

$$\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \geq d \qquad \forall t \geq 0 \qquad \Longleftrightarrow$$

$$f_{ij}(t) \; =: \; \|\mathbf{v}_{ij}\|^2 \; t^2 + 2(\mathbf{x}_{ij}^0 \cdot \mathbf{v}_{ij}) \; t + (\|\mathbf{x}_{ij}^0\|^2 - d^2) \geq 0 \qquad \forall t \geq 0$$

since $\mathbf{x}_i(t) - \mathbf{x}_j(t) = \mathbf{x}_{ij}^0 + \mathbf{v}_{ij}t$,

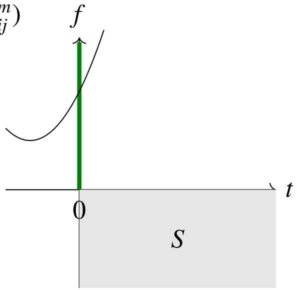
where:

$\mathbf{x}_{ij}^0$ = initial relative position of aircraft i and j $\quad (= \mathbf{x}_i^0 - \mathbf{x}_j^0)$

$\mathbf{v}_{ij}$ = relative speed of aircraft i and j

strictly convex univariate quadratic function!

minimized at $t_{ij}^m := -\frac{(\mathbf{x}_{ij}^0 \cdot \mathbf{v}_{ij})}{\|\mathbf{v}_{ij}\|^2}$ with value $f_{ij}^m := f_{ij}(t_{ij}^m)$



Separation: $t_{ij}^m \leq 0 \quad \text{or} \quad f_{ij}^m \geq 0$

$S := \{(t, f) : t > 0 \text{ and } f \leq 0\}$

[S. Cafieri, N. Durand, JOGO 2014]

Inter-distance safety separation

$$\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \geq d \qquad \forall t \geq 0 \qquad \Longleftrightarrow$$

$$f_{ij}(t) \;=:\; \|\mathbf{v}_{ij}\|^2\, t^2 + 2(\mathbf{x}_{ij}^0 \cdot \mathbf{v}_{ij})\, t + (\|\mathbf{x}_{ij}^0\|^2 - d^2) \geq 0 \qquad \forall t \geq 0$$

since $\mathbf{x}_i(t) - \mathbf{x}_j(t) = \mathbf{x}_{ij}^0 + \mathbf{v}_{ij} t$,

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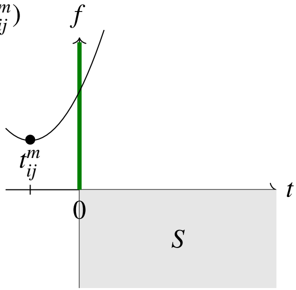
Separation: $t_{ij}^m \leq 0 \quad$ or $\quad f_{ij}^m \geq 0$



$S := \{(t, f) : t > 0 \text{ and } f < 0\}$

[S. Cafieri, N. Durand, JOGO 2014]

Inter-distance safety separation

$$\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \geq d \qquad \forall t \geq 0$$

$$\Longleftrightarrow$$

$$f_{ij}(t) \; =: \; \|\mathbf{v}_{ij}\|^2 \, t^2 + 2(\mathbf{x}^0_{ij} \cdot \mathbf{v}_{ij})\, t + (\|\mathbf{x}^0_{ij}\|^2 - d^2) \geq 0 \qquad \forall t \geq 0$$

since $\mathbf{x}_i(t) - \mathbf{x}_j(t) = \mathbf{x}^0_{ij} + \mathbf{v}_{ij} t$,

where:

$\mathbf{x}^0_{ij}$ = initial relative position of aircraft i and j $\quad (= \mathbf{x}^0_i - \mathbf{x}^0_j)$

$\mathbf{v}_{ij}$ = relative speed of aircraft i and j

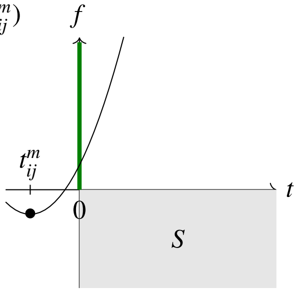
strictly convex univariate quadratic function!

minimized at $t^m_{ij} := -\frac{(\mathbf{x}^0_{ij} \cdot \mathbf{v}_{ij})}{\|\mathbf{v}_{ij}\|^2}$ with value $f^m_{ij} := f_{ij}(t^m_{ij})$

Separation: $t^m_{ij} \leq 0 \quad \text{or} \quad f^m_{ij} \geq 0$

as long as $f_{ij}(0) \geq 0$

(assume initially separated!)



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[S. Cafieri, N. Durand, JOGO 2014]

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$$f_{ij}(t) \;=:\; \|\mathbf{v}_{ij}\|^2\, t^2 + 2(\mathbf{x}^0_{ij} \cdot \mathbf{v}_{ij})\, t + (\|\mathbf{x}^0_{ij}\|^2 - d^2) \geq 0 \qquad \forall t \geq 0$$

since $\mathbf{x}_i(t) - \mathbf{x}_j(t) = \mathbf{x}^0_{ij} + \mathbf{v}_{ij} t$,

where:

$\mathbf{x}^0_{ij}$ = initial relative position of aircraft i and j $\quad (= \mathbf{x}^0_i - \mathbf{x}^0_j)$

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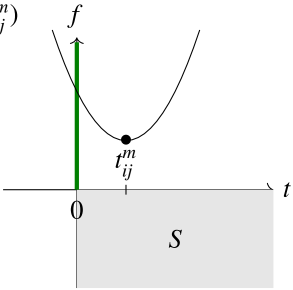
minimized at $t^m_{ij} := -\frac{(\mathbf{x}^0_{ij} \cdot \mathbf{v}_{ij})}{\|\mathbf{v}_{ij}\|^2}$ with value $f^m_{ij} := f_{ij}(t^m_{ij})$

Separation: $t^m_{ij} \leq 0$ or $f^m_{ij} \geq 0$



$S := \{(t, f) : t > 0 \text{ and } f \leq 0\}$

[S. Cafieri, N. Durand, JOGO 2014]

Inter-distance safety separation

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$$f_{ij}(t) \;=:\; \|\mathbf{v}_{ij}\|^2\, t^2 + 2(\mathbf{x}^0_{ij}\cdot \mathbf{v}_{ij})\, t + (\|\mathbf{x}^0_{ij}\|^2 - d^2) \geq 0 \qquad \forall t \geq 0$$

since $\mathbf{x}_i(t) - \mathbf{x}_j(t) = \mathbf{x}^0_{ij} + \mathbf{v}_{ij} t$,
where:

$\mathbf{x}^0_{ij}$ = initial relative position of aircraft i and j $\quad (= \mathbf{x}^0_i - \mathbf{x}^0_j)$

$\mathbf{v}_{ij}$ = relative speed of aircraft i and j

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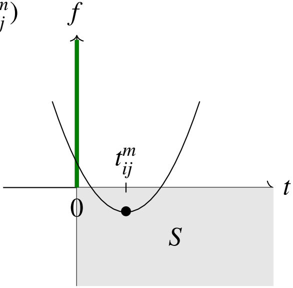
minimized at $t^m_{ij} := -\frac{(\mathbf{x}^0_{ij}\cdot\mathbf{v}_{ij})}{\|\mathbf{v}_{ij}\|^2}$ with value $f^m_{ij} := f_{ij}(t^m_{ij})$



Separation: $t^m_{ij} \leq 0 \quad \text{or} \quad f^m_{ij} \geq 0$

[S. Cafieri, N. Durand, JOGO 2014]

$S := \{(t,f) : t > 0 \text{ and } f \leq 0\}$

New formulations [S.C., A.R. Conn, M. Mongeau, EJOR 2023]

Deciding for each aircraft $i \in A$:

- heading angle deviation $\phi_i \rightarrow \phi_i + \theta_i$
- speed deviation $v_i \rightarrow q_i v_i$

$$\mathbf{v}_{ij} = \begin{pmatrix} cos(\overbrace{\phi_i + \theta_i}^{c_i})q_i v_i - cos(\phi_j + \theta_j)q_j v_j \\ sin(\underbrace{\phi_i + \theta_i}_{s_i})q_i v_i - sin(\phi_j + \theta_j)q_j v_j \end{pmatrix}$$

Considering rather (**reformulation** to avoid trigonometric functions):

$$\omega_i := c_i q_i v_i \qquad \mathbf{v}_{ij} = \begin{pmatrix} \omega_i - \omega_j \\ \pi_i - \pi_j \end{pmatrix}, \qquad i,j \in A : i < j$$

$$\pi_i := s_i q_i v_i$$

The constraints to be satisfied are:

$$t_{ij}^m \leq 0 \quad \text{or} \quad f_{ij}^m \geq 0 \qquad i,j \in A : i < j$$

$$f_{ij}^m \|\mathbf{v}_{ij}\|^2 = \|\mathbf{v}_{ij}\|^2 (\|\mathbf{x}_{ij}^0\|^2 - d^2) - (\mathbf{x}_{ij}^0 \cdot \mathbf{v}_{ij})^2 \qquad i,j \in A : i < j$$

$$t_{ij}^m \|\mathbf{v}_{ij}\|^2 = -\mathbf{x}_{ij}^0 \cdot \mathbf{v}_{ij} \qquad i,j \in A : i < j$$

$$\mathbf{v}_{ij} = \begin{pmatrix} \omega_i - \omega_j \\ \pi_i - \pi_j \end{pmatrix} \qquad i,j \in A : i < j$$

$$\omega_i^2 + \pi_i^2 = (q_i v_i)^2 \qquad i \in A$$

$$\underline{q_i} \leq q_i \leq \overline{q_i} \qquad i \in A$$

$$\underline{\omega_i} \leq \omega_i \leq \overline{\omega_i}, \quad \underline{\pi_i} \leq \pi_i \leq \overline{\pi_i} \qquad i \in A$$

MINLP formulation

$$\min_{\omega,\pi,q,z,\mathbf{v},b} \quad (1-\lambda)\sum_{i\in A}(q_i-1)^2 + \lambda\sum_{i\in A} b_i$$

s.t.

$$z_{ij}\left(\|\mathbf{v}_{ij}\|^2\left(\|\mathbf{x}^0_{ij}\|^2 - d^2\right) - (\mathbf{x}^0_{ij}\cdot\mathbf{v}_{ij})^2\right) \geq 0 \qquad i,j\in A: i<j$$

$$(-\mathbf{x}^0_{ij}\cdot\mathbf{v}_{ij})(z_{ij}-1)\geq 0 \qquad i,j\in A: i<j$$

$$\mathbf{v}_{ij} = \begin{pmatrix}\omega_i-\omega_j\\ \pi_i-\pi_j\end{pmatrix} \qquad i,j\in A: i<j$$

$$\omega_i^{\,2}+\pi_i^{\,2} = (q_i v_i)^2 \qquad i\in A$$

$$\omega_i \geq \left(\min\left(\underline{a}\cos\phi_i,\cos\phi_i\right) - b_i\,|\sin\phi_i|\right) q_i v_i \qquad i\in A$$

$$\omega_i \leq \left(\max\left(\underline{a}\cos\phi_i,\cos\phi_i\right) + b_i\,|\sin\phi_i|\right) q_i v_i \qquad i\in A$$

$$\pi_i \geq \left(\min\left(\underline{a}\sin\phi_i,\sin\phi_i\right) - b_i\,|\cos\phi_i|\right) q_i v_i \qquad i\in A$$

$$\pi_i \leq \left(\max\left(\underline{a}\sin\phi_i,\sin\phi_i\right) + b_i\,|\cos\phi_i|\right) q_i v_i \qquad i\in A$$

$$\underline{q}_i \leq q_i \leq \overline{q}_i \qquad i\in A$$

$$\underline{\omega}_i \leq \omega_i \leq \overline{\omega}_i \qquad i\in A$$

$$\underline{\pi}_i \leq \pi_i \leq \overline{\pi}_i \qquad i\in A$$

$$0 \leq b_i \leq \overline{b} \qquad i\in A$$

$$z_{ij}\in\{0,1\} \qquad i,j\in A: i<j$$

minimizing speed and angle deviations

$0\leq\lambda\leq 1$

b_i s.t., $\forall\, i\in A$:

$-\bar{b} \leq -b_i \leq \sin(\theta_i) \leq b_i$

(minimize b_i to minimize angle deviations)

Formulation based on the penalty function g_β

Penalize the logical constraints in the objective:

$$\sum_{i,j \in A: i<j} g_\beta(t_{ij}^m, f_{ij}^m)$$

(keeping the other constraints)

- (nonconvex) NLP
- potential *local infeasibility*: we implement a simple multistart heuristic

$$\min_{q,\omega,\pi,t^m,f^m,\mathbf{v}} \sum_{1 \le i<j \le n} g_\beta(t_{ij}^m, f_{ij}^m)$$

s.t.

$$f_{ij}^m \|\mathbf{v}_{ij}\|^2 = \|\mathbf{v}_{ij}\|^2 \left(\|\mathbf{x}_{ij}^0\|^2 - d^2 \right) - (\mathbf{x}_{ij}^0 \cdot \mathbf{v}_{ij})^2 \qquad i,j \in A : i<j$$

$$t_{ij}^m \|\mathbf{v}_{ij}\|^2 = -\mathbf{x}_{ij}^0 \cdot \mathbf{v}_{ij} \qquad i,j \in A : i<j$$

$$\mathbf{v}_{ij} = \begin{pmatrix} \omega_i - \omega_j \\ \pi_i - \pi_j \end{pmatrix} \qquad i,j \in A : i<j$$

$$\omega_i{}^2 + \pi_i{}^2 = (q_i v_i)^2 \qquad i \in A$$

$$\underline{q_i} \le q_i \le \overline{q_i} \qquad i \in A$$

$$\underline{\omega}_i \le \omega_i \le \overline{\omega}_i \qquad i \in A$$

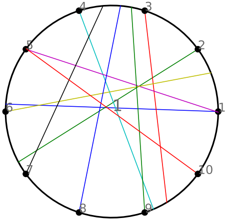
$$\underline{\pi}_i \le \pi_i \le \overline{\pi}_i \qquad i \in A$$

Numerical results

AMPL model implementation, NLP solver: `IPOPT 3.12`
2.66 GHz, 32 GB RAM

Random Circle Problem (RCP)

[Rey & Hijazi, 2017]



- n=10, 20, 30 aircraft
- d= 5 NM

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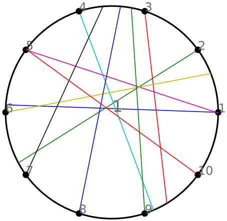

- n=10, 20, 30 aircraft
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NLP penalty: on all the 35 instances

- solutions satisfying the separation constraints (**zero-value penalty function**)
- only 2 starting points needed for the multistart heuristic (and only for 2 instances)

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AMPL model implementation, NLP solver: `IPOPT 3.12`
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Random Circle Problem (RCP)

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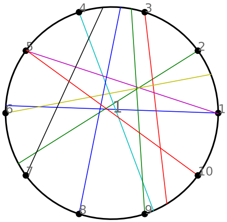


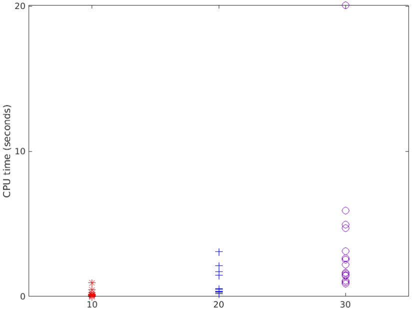
- n=10, 20, 30 aircraft
- d= 5 NM

NLP penalty: on all the 35 instances

- solutions satisfying the separation constraints (**zero-value penalty function**)
- only 2 starting points needed for the multistart heuristic (and only for 2 instances)
- CPU times: $\mu = 3.68$ seconds for $n = 30$

Time (s)

n	mean	st.dev.	min	max
10	0.23	0.28	0.00	0.94
20	1.05	0.99	0.19	3.09
30	3.68	4.79	0.90	20.06



3-phase algorithm

Initialize: $q^c := 1, \ \omega^c, \pi^c$ such that $\theta = 0$, upper_bound $:= +\infty$

(1) NLP penalty:

- solve by local optimization
- if (zero penalty): feasible (conflict-free) solution $q^c, \omega^c, \pi^c \ \Rightarrow \ b^c, z^c$
- compute $q_{dev} := \sum_{i \in A}(1 - q_i^c)^2$ **if** $q_{dev} \leq tol$ **then** Stop

3-phase algorithm

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(2) $MINLP_{fix}$ with fixed $z = z^c$:

- solve by continuous global optimization
  starting from $(q^c, \omega^c, \pi^c, b^c, z^c)$ with $\lambda = 0$ (minimizing speed deviation)
  using upper_bound=q_{dev} as cutoff
  to get new $(q^c, \omega^c, \pi^c, b^c, z^c)$
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(2) $MINLP_{fix}$ with fixed $z = z^c$:

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  to get new $(q^c, \omega^c, \pi^c, b^c, z^c)$
- compute $q_{dev} := \sum_{i \in A}(1 - q_i^c)^2$ **if** $q_{dev} \leq tol$ **then** Stop

(3) MINLP: (free z)

- solve by mixed-integer global optimization
  starting from the last computed solution
  with $\lambda = 0$ (minimizing speed deviation)

Results: 3-phase algorithm

AMPL model implementation
MINLP solver: `COUENNE 0.5`, NLP solver: `IPOPT 3.12`

$tol = 10^{-7}$
MINLP alone reaches tlim=600 sec. on 60% instances with $n = 30$

Name	n	n_c	n_{hth}	2nd phase: MINLP_{fix} time (s)	2nd phase: MINLP_{fix} speed dev.	3rd phase: MINLP time (s)	3rd phase: MINLP speed dev.	Total time (s)
RCP_30_1	30	35	1	34.93	1.4e-06	7.900	4.22e-17	45.94
RCP_30_2	30	38	1	59.69	6.5e-15	–	–	60.76
RCP_30_3	30	46	1	2.920	1.7e-16	–	–	3.816
RCP_30_4	30	39	1	62.98	2.6e-16	–	–	65.60
RCP_30_5	30	36	2	43.98	2.2e-06	tlim	2.24e-06	tlim
RCP_30_6	30	32	2	43.99	1.5e-16	–	–	46.20
RCP_30_7	30	18	1	105.2	1.2e-16	–	–	106.8
RCP_30_8	30	40	1	2.700	1.3e-17	–	–	4.124
RCP_30_9	30	41	2	51.23	3.3e-06	20.38	1.89e-17	76.56
RCP_30_10	30	46	1	67.59	7.3e-07	–	–	73.46
RCP_30_11	30	34	2	51.79	4.4e-16	–	–	52.79
RCP_30_12	30	36	1	86.90	4.8e-15	–	–	88.45
RCP_30_13	30	30	1	4.092	3.6e-16	–	–	6.632
RCP_30_14	30	39	2	3.752	8.8e-18	–	–	23.81
RCP_30_15	30	30	1	79.32	9.9e-16	–	–	84.05

Outline

1 Continuous penalty-based formulation of logical constraints

2 An application from discrete geometry

3 An application from Air Traffic Management

4 Conclusions and perspectives

Conclusions and perspectives

Summary of contributions

- For nonlinear problems whose discrete nature arise from logical constraints: a **continuous-optimization alternative** to compute **good-quality upper bounds**
- **Usefulness to efficiently compute global solutions** demonstrated on two applications from different domains

Perspectives

Promising to address **other problems** involving logical constraints that would incur too numerous extra binary variables

S. Cafieri, A. R. Conn, and M. Mongeau.
The continuous quadrant penalty formulation of logical constraints.
Open Journal on Mathematical Optimization, 2023.

S. Cafieri, A. R. Conn, and M. Mongeau.
Mixed-integer nonlinear and continuous optimization formulations for aircraft conflict avoidance via heading and speed deviations.
European Journal of Operational Research, 2023.

S. Cafieri, P. Hansen, and F. Messine.
Global exact optimization for covering a rectangle with 6 circles.
Journal of Global Optimization, 2022.