Maximisation de trajectoires de systèmes dynamiques: approximations et perturbations

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Example From Real-Life / Lane Keeping System

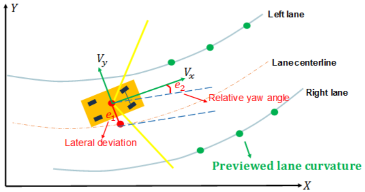

Figure: Mathworks/MPC Toolbox/ Automated Driving

A. Lane Keeping Setup

Similar to [1], we consider a time-discretized version of the vehicle model in [27] (using the forward Euler method with sampling time t_s):

$$x'_{k+1} = (I + At_s)x'_k + Bt_s u_k + Et_s r_{d,k}, \qquad (19)$$

where

$$A = \begin{bmatrix} 0 & 1 & V_0 & 0 \\ 0 & -\frac{C_f + C_r}{MV_0} & 0 & \frac{bC_r - aC_f}{MV_0} - V_0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{bC_r - aC_f}{I_z V_0} & 0 & -\frac{a^2 C_f + b^2 C_r}{I_z V_0} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{C_f}{M} \\ 0 \\ a\frac{C_f}{I_z} \end{bmatrix}, E = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

The states $x'_k = \begin{bmatrix} y_k & \nu_k & \psi_k & r_k \end{bmatrix}^\top$ are the lateral displacement of the car from the center of the lane (y_k), the car's lateral velocity (ν_k), the yaw angle of the car with respect to the lane center (ψ_k), and the yaw rate of the car (r_k). The input u_k is the angle of the front tires at the current time step k. Road curvature is modeled as a *known* disturbance to the system, and the road curves at a rate of $r_{d,k} = \frac{V_0}{R_k}$ where V_0 is the longitudinal velocity of the vehicle and R_k

Reference:

Khajenejad et al. **Tractable Compositions of Discrete-Time Control Barrier Functions with Application to Lane Keeping and Obstacle Avoidance.** ECC 2021.

2) ***Lane Centering Constraint:*** This second constraint keeps the car from drifting too far away from the middle of the lane, where it could possibly drift out of it. This can be done by restricting the maximum lateral displacement:

$$|y_k| \leq y_{max}, \ \forall k \in \mathbb{N}. \qquad (21)$$

As described in [1], a typical United States lane is 12 feet wide while a car is about 6 feet wide, so the maximum lateral displacement the car can safely experience is 3 feet to either side, so $y_{max} = 3$ feet ≈ 0.9 meters.

Academic Example / Peaks Computation Problem

II. Motivating Examples

Consider now two examples of estimating the norms of powers of matrices; they allow for closed-form expressions and demonstrate that the magnitude of peak may be arbitrarily high.

Given a Schur stable matrix A, we are targeted at estimating

$$\eta(A) = \max_{k=1,2,\ldots} \|A^k\|;$$

if $\eta(A) > 1$ and the maximum is reached at $k > 1$, we say that peak of the quantity $\|A^k\|$ takes place, and we want to find the peak instant

$$k^* = \arg \max_{k=1,2,\ldots} \|A^k\|.$$

We consider linear discrete-time systems of the form

$$x_{k+1} = Ax_k, \quad x_k \in \mathbb{R}^n, \; k = 0, 1, \ldots \tag{3}$$

with initial conditions x_0, $\|x_0\|_2 \leq 1$ (here we use the Euclidian $\|\cdot\|_2$ norm for vectors and the spectral norm $\|A\|_2$ for matrices). For the class of Schur stable systems we present a simple upper bound on the peak of their *trajectories*:

$$\max_{\|x_0\|_2 \leq 1} \max_{k=0,1,\ldots} \|x_k\|_2.$$

Reference:

Shcherbakov and Parsegov. **Solutions of Discrete Time Linear Systems: Upper Bounds on Deviations.** International Conference on System Theory, Control and Computing 2018.

General Mathematical Model of Verification

Reachable values set of a discrete-time dynamical system

Let us consider $S := (X^{\text{in}}, T)$ where:

- initial conditions $X^{\text{in}} \subset \mathbb{R}^d$ (compact);
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and, for S, its trajectories and **its reachable values set** $\mathfrak{R}$ i.e.

- $x_0 \in X^{\text{in}}$, and $\forall\, k \in \mathbb{N}$, $x_{k+1} := T(x_k)$;
- $\mathfrak{R} := \bigcup_{k \geq 0} T^k(X^{\text{in}})$

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Invariant Properties

A set $P \subset \mathbb{R}^d$ is said to be an **invariant** of S if and only if $\mathfrak{R} \subseteq P$.

ex : lane keeping for the vehicle.

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Verification Problem

Given $S = (X^{\mathrm{in}}, T)$ and $P \subset \mathbb{R}^d$, **prove that P is an invariant for S without any simulation.**

Optimization Formulation

Assume that $P := \{x \in \mathbb{R}^d \mid \varphi(x) \leq \alpha\}$ where $\varphi : \mathbb{R}^d \to \mathbb{R}$ and $\alpha \in \mathbb{R}$.

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Let us consider the maximization problem:

$$\operatorname*{Max}_{x \in \mathfrak{R}} \varphi(x)$$

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Theorem

1. *To prove the property, a* **upper bound less than** α *suffices;*
2. *To disprove the property, a* **feasible solution** $\overline{x}$ **s.t.** $\varphi(\overline{x}) > \alpha$ *suffices.*

Outline

1. Approximation
2. Exact Optimal Solutions
3. Perturbations

Example: Sublevel Region Avoidance

Let us consider the following dynamical system:

- $x^0 \in [0.5, 0.7] \times [0.5, 0.7]$;
- for all $k \geq 0$, $x^{k+1} = T(x^k)$, where:

$$T(x) = \begin{cases} \begin{pmatrix} x_1^2 + x_2^3 \\ x_1^3 + x_2^2 \end{pmatrix} & \text{if } \|x\|_2^2 - 1 < 0 \\ \begin{pmatrix} 0.5x_1^3 + 0.4x_2^2 \\ -0.6x_1^2 + 0.3x_2^2 \end{pmatrix} & \text{otherwise} \end{cases}$$

Question:

Let B be the disk centered at $(-0.5, -0, 5)$ of radius 0.5.

Can we prove automatically that the property "$\forall\, k \in \mathbb{N},\ x^k \notin B$" is true?

Example : Sublevel Region Avoidance

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$$T(x) = \begin{cases} T_1(x) & \text{if } \|x\|_2^2 < 1 \\ T_2(x) & \text{otherwise} \end{cases}$$

where:

$$T_1(x) = \begin{pmatrix} x_1^2 + x_2^3 \\ x_1^3 + x_2^2 \end{pmatrix}$$

and

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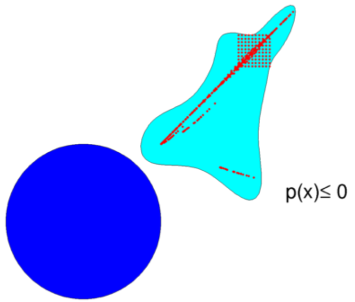

A discretization of $\{x^k, k \in \mathbb{N}\}$ in red, a proof certificate in cyan and B in blue.

Formulation

Key Tools:

$$\mathfrak{R} = \min\{X \mid F(X) \subseteq X\} \text{ where } C \mapsto F(C) = T(C) \cup X^{\mathrm{in}} \ .$$

Then:

$$\sup_{x\in\mathfrak{R}} \varphi(x) = \inf_{\{C\subset\mathbb{R}^d : F(C)\subseteq C\}} \sup_{x\in C} \varphi(x) \leq \inf_{C:\begin{cases} F(C)\subseteq C \\ C = p^{-1}(\mathbb{R}_-) \\ p \text{ polynomial}\end{cases}} \sup_{x\in C} \varphi(x) \ .$$

To solve the optimization on polynomials (φ is polynomial):

1. Linearize for $C = p^{-1}(\mathbb{R}_-)$:
   - (reinforcement) $F(C) \subseteq C$;
   - (upper-bound) $\sup_C \varphi$;
2. Reinforce the linearized problem using Sums-Of-Squares (SOS);
3. Fix the maximal degree of p.

Challenges

Few mathematical questions arise:

1. If $\sup_{x\in\mathfrak{R}} \varphi(x) < +\infty$, can we guarantee the existence of p s.t.

$$\mathfrak{R} \subseteq p^{-1}(\mathbb{R}_-) \text{ and } \sup_{p(x)\leq 0} \varphi(x) < +\infty?$$

2. Is there p s.t.

$$\mathfrak{R} \subseteq p^{-1}(\mathbb{R}_-) \text{ and } \sup_{p(x)\leq 0} \varphi(x) = \sup_{x\in\mathfrak{R}} \varphi(x)?$$

3. Have we at least

$$\lim_{m\to+\infty} \inf_{p:\begin{cases}\mathfrak{R}\subseteq p^{-1}(\mathbb{R}_-)\\ \deg(p)\leq m\end{cases}} \sup_{p(x)\leq 0} \varphi(x) = \sup_{x\in\mathfrak{R}} \varphi(x)?$$

Outline

Exact approaches

Let :

$$\nu_k := \sup_{x \in X^{\mathrm{in}}} \varphi(T^k(x)) \text{ and } \nu_{\mathrm{opt}} := \sup_{k \in \mathbb{N}} \nu_k.$$

$\Longrightarrow \nu_{\mathrm{opt}} = \sup_{y \in \mathfrak{R}} \varphi(y).$

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To solve $\underset{y \in \mathfrak{R}}{\mathrm{Max}}\, \varphi(y)$, we compute K such that:

$$\sup_{k \in \mathbb{N}} \sup_{x \in X^{\mathrm{in}}} \varphi(T^k(x)) = \max_{k=0,\ldots,K} \sup_{x \in X^{\mathrm{in}}} \varphi(T^k(x))$$

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Proposition

If there exists $k \in \mathbb{N}$ s.t. $\nu_k > \limsup_{n \to +\infty} \nu_n$, then such a K exists.

Using Homeorphisms and Geometric Sequences

Let $u \in \mathbb{R}^{\mathbb{N}}$ and $\mathbf{K}_u^{>} := \max\{k \in \mathbb{N} : u_k = \max_n u_n\}$.

Theorem (Optimal Bijection)

Suppose that $u_k > \limsup_n u_n$. *Then there exists* $(a, b, c) \in \mathbb{R}_+^* \times (0, 1) \times \mathbb{R}$ *:*
(1) $\forall\, k \in \mathbb{N},\ u_k \leq ab^k + c$; *(2)* $\exists\, k \in \mathbb{N}$ *s.t.* $u_k > c$; *(3)* $u_{\mathbf{K}_u^{>}} = ab^{\mathbf{K}_u^{>}} + c$.

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Corollary

Let us write $I = (0,1)$ *and let:*

$$\Gamma(u) := \left\{ (h, \beta) : \begin{array}{c} h : I \mapsto h(I) \text{ strictly increasing and continuous,} \\ \beta \in I \text{ and } \forall\, k \in \mathbb{N},\ u_k \leq h(\beta^k) \end{array} \right\}$$

Then :

$$\mathbf{K}_u^> = \min_{(h,\beta) \in \Gamma(u)} \min_{u_k > h(0)} \frac{\ln(h^{-1}(u_k))}{\ln(\beta)}$$

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How to compute an element of $\Gamma(\nu)$ for our initial sequence ν?

Linear Systems

When :

- the dynamics $T \sim A$ linear and stable (i.e. $\rho(A) < 1$) and X^{in} polytope;
- the objective function φ is quadratic $\varphi \sim Q$,

(h, β) can be computed from a quadratic Lyapunov function$\sim P$ s.t.:

$$P \succ 0, \; P - A^\intercal PA \succ 0$$

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If $\nu_k > 0$ for some k, we have :

$$\max_{k \in \mathbb{N}} \sup_{x \in X^{\mathrm{in}}} x^{\intercal} A^{\intercal} Q A x = \max_{k=0,\ldots,K} \sup_{x \in X^{\mathrm{in}}} x^{\intercal} A^{\intercal} Q A x$$

where :

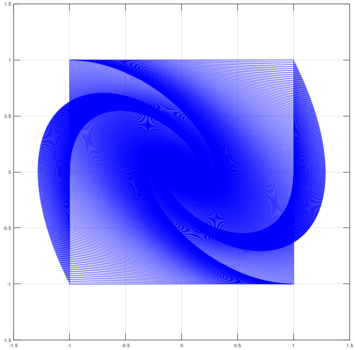
$$K = \left\lfloor \frac{\ln\left(\frac{\nu_{\mathrm{opt}}}{t^* \mu(P)}\right)}{2 \ln\left(\|A\|_P\right)} \right\rfloor + 1 \text{ where } \begin{cases} t^* = \lambda_{\max}\left(QP^{-1}\right) \\ \mu(P) = \max_{x \in X^{\mathrm{in}}} x^{\intercal} P x \\ \|A\|_P = \sqrt{\lambda_{\max}\left(A^{\intercal} P A P^{-1}\right)} \end{cases}$$

A Toy Example

Let us consider a linear system (an Euler scheme of an EDO) in $\mathbb{R}^2$:

$$\begin{pmatrix} x_{k+1} \\ v_{k+1} \end{pmatrix} = A \begin{pmatrix} x_k \\ v_k \end{pmatrix} = \begin{pmatrix} 1 & 0.01 \\ -0.01 & 0.99 \end{pmatrix} \begin{pmatrix} x_k \\ v_k \end{pmatrix}, \quad (x_0, v_0) \in [-1, 1]^2$$

$\mathfrak{R}$ of $(A, [-1, 1]^2))$:



$$\operatorname*{Max}_{(x,v)\in\mathfrak{R}} x^2 \implies Q = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$$

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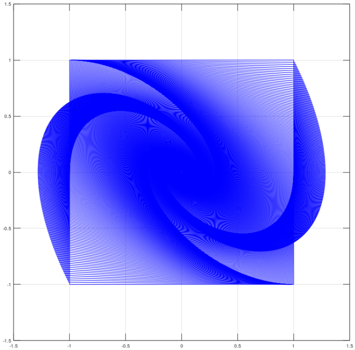

Solve $P - A^\intercal P A \succ 0$, we get:

$$\overline{P} = \begin{pmatrix} 4.9501 & 1.5475 \\ 1.5475 & 4.5313 \end{pmatrix}$$

Hence:

- $t_Q^*\left(\overline{P}\right) = \lambda_{\max}\left(QP^{-1}\right) \simeq 0.226165$;
- $\mu(\overline{P}) = \max_{y \in [-1,1]^2} y^\intercal P y \simeq 12.57646$;
- $\|A\|_{\overline{P}}^2 = \lambda_{\max}\left(A^\intercal P A P^{-1}\right) \simeq 0.993853$.

Finally:

$$K = 89 \implies k_{\text{opt}} = 61 \text{ and } \nu_{\text{opt}} \simeq 1.64886.$$

Going beyond

Theorem (First-Order Condition)

The functions $P \mapsto \dfrac{\ln\left(\frac{\nu_k}{t^*\mu(P)}\right)}{2\ln\left(\|A\|_P\right)}$ *are Hadamard/Frechet semidifferentiable.*

We can deduce a first-order condition.

Going beyond

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$\Longrightarrow$ Nondifferentiable (nonlinear) semidefinite programming?

Going beyond

Theorem (General systems $S = (X^{\text{in}}, T)$)

The two statements are equivalent:

1. *There exists* $(h, \beta) \in \Gamma(\nu)$ *such that* $\{k \in \mathbb{N} : \nu_k > h(0)\} \neq \emptyset$;
2. *there exists a "Lyapunov function"* $V : \mathbb{R}^d \mapsto [0, +\infty]$ *compatible with* φ:
   - $\sup_{X^{\text{in}}} V \in (0, 1]$;
   - $\lambda V - V \circ T \geq 0$ *for some* $\lambda \in (0, 1)$

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$\Longrightarrow$ Computations of such a V?

Going beyond

- Nondifferentiable (nonlinear) semidefinite programming.
- Computations of general "Lyapunov function" V.

Outline

1. Approximation
2. Exact Optimal Solutions
3. Perturbations

Considering Nondeterminism and Randomness of entries

Nonautonomous discrete-time systems (e.g. lane keeping systems) need external entries (from sensor values) which could generate deviations on ideal values.

$\Longrightarrow$ **how to handle those deviations in problem verifications?**

Dynamical system with entries

A dynamical system $S = (X^{\text{in}}, X^{\infty}, E, T)$ with entries is represented by four data:

1. An initial set X^{in} which a compact subset of $\mathbb{R}^d$;
2. An live set X^{∞};
3. An entries set E supposed to be a compact set;
4. A dynamic T which is a function from $\mathbb{R}^d \times E$ to $\mathbb{R}^d$ i.e. $T : \mathbb{R}^d \times E \mapsto \mathbb{R}^d$.

The evolution of the state starting from $x_0 \in X^{\text{in}}$ follows, for all $k \in \mathbb{N}$:

$$x_{k+1} = \begin{cases} T(x_k, e_k) & \text{if } x_k \in X^{\infty} \\ x_k & \text{otherwise} \end{cases}$$

where $e_k \in E$ for all $k \in \mathbb{N}$.

Perturbed dynamical system

A perturbed dynamical system $\widehat{S}$ of $S = (X^{\mathrm{in}}, X^{\infty}, E, T)$ is characterized by :

1. An *initial set* which is X^{in};
2. An invariant set which is X^{∞};
3. An entries set which is E;
4. A pertubations set U supposed to be a compact subset of a topological vector space;
5. A perturbed *dynamic* $\widehat{T} : \mathbb{R}^d \times E \times (U \cup \{0\}) \mapsto \mathbb{R}^d$ s.t.

$$\widehat{T}(x, e, 0) = T(x, e)$$

The evolution of the state starting from $x_0 \in X^{\mathrm{in}}$ follows, for all $k \in \mathbb{N}$:

$$x_{k+1} = \begin{cases} \widehat{T}(x_k, e_k, u_k) & \text{if } x_k \in X^{\infty} \\ x_k & \text{otherwise} \end{cases}$$

where $e_k \in E$ and $u_k \in U$.

Robustness problem

Introducing for $e \in E$:

$$T_e : \mathbb{R}^d \ni x \mapsto T_e(x) := T(x, e) \text{ and } \mathop{\circ}_{k=0}^{K} T_{e_k} := T_{e_K} \circ T_{e_{K-1}} \circ \cdots T_{e_1} \circ T_{e_0}.$$

We can define the reachable value set of $S = (X^{\text{in}}, X^{\infty}, E, T)$:

$$\mathfrak{R}_S = X^{\text{in}} \cup \bigcup_{k \in \mathbb{N}} \bigcup_{(e_0, \dots, e_{k-1}) \in E^k} \mathop{\circ}_{k=0}^{K} T_{e_k}(X^{\text{in}})$$

We can define similarly $\mathfrak{R}_{\hat{S}}$ taking $E' = (E \times U)$.

An invariant P is robust against the perturbed system $\hat{S}$ associated to S iff

$$\mathfrak{R}_S \subset P \implies \mathfrak{R}_{\hat{S}} \subset P$$

Model on Nondeterminism and Randomness

In this context, we have:

- Non-determinism $e_k \to$ finite family of closed subsets $(A_i)_{i \in I}$;
- Randomness $\to e_k \in A_i$ according to some probability μ;
- Non-determinism on $\mu(A_i) \to$ values of $\mu(A_i)$ not precisely known $\to$ finite family of $\{(\underline{\alpha_i}, \overline{\alpha_i})\}_{i \in I}$.

Our imprecise probabilities are modeled as a convex family of probability measures and we can define :

$$\Gamma := \{\mu \text{ proba } : 0 \leq \underline{\alpha_i} \leq \mu(A_i) \leq \overline{\alpha_i} \leq 1, \forall i \in I, supp(\mu) \subseteq U\}$$

Then we can define the trajectories on the states of the perturbed system.

$$X_{k+1} = \begin{cases} \widehat{T}(X_k, e_k, \xi) & \text{if } X_k \in X^\infty \\ X_k & \text{otherwise} \end{cases}$$

where the law of ξ is not perfectly known but belongs to Γ.

Non-robustness

Let us define:

$$\Gamma_m = \left\{ \mu \text{ proba} \;\middle|\; \begin{array}{l} \text{supp}(\mu) \text{ finite }, \forall A \in \mathcal{B}(U),\ \mu(A) = \displaystyle\sum_{x \in \text{supp}(\mu) \cap A} \mu(\{x\}), \\ \text{card}\,(\text{supp}(\mu) \cap A_i) = m \\ \underline{\alpha_i} \leq \mu(A_i) \leq \overline{\alpha_i}, \mu(U) = 1 \end{array} \right\}$$

Theorem (Non-robustness)

If

1. $P = \varphi^{-1}(-\infty, \beta]$;
2. $\varphi \circ \circ_{k=0}^{K} \hat{T}_{e_k,\xi}$ *is measurable and integrable;*
3. *and*

$$\sup_{\mu \in \Gamma_1} \sup_{x \in X^{\text{in}}} \sup_{K \in \mathbb{N}} \sup_{(e_0,\ldots,e_{K-1}) \in E^K} \mathbb{E}_\mu \left(\varphi\left(\mathop{\circ}_{k=0}^{K} \hat{T}_{e_k,\xi}(x) \right) \right) > \beta$$

then P is not robust.

Computations

Efficient computations of

$$\sup_{\mu \in \Gamma_1} \sup_{x \in X^{\mathrm{in}}} \sup_{K \in \mathbb{N}} \sup_{(e_0, \ldots, e_{K-1}) \in E^K} \mathbb{E}_\mu \left(\varphi(\mathop{\circ}_{k=0}^{K} \hat{T}_{e_k, \xi}(x)) \right)$$

in simple cases i.e.

1. The perturbed system is affine and stable;
2. The sets E and X^{in} are polytopes;
3. The sets A_i are also polytopes;
4. The function φ is linear

?